Projective algebraic sets

A point $P \in \mathbb{P}^n$ is a *zero* of a polynomial $f \in k[x_1, \dots, x_{n+1}]$ if $f(a_1, \dots, a_{n+1}) = 0$ for every choice $P = [a_1 : a_2 : \dots : a_{n+1}]$ of homogeneous coordinates for P.

Ex: $f(x,y) = x - y + 1$. Then if $P = [2:1] = [4:2]$, $f(2,1) = 0$, but $f(4,2) \neq 0$, so P is not a zero of f.

Claim: If F is homogeneous and vanishes at one representative of P, then it vanishes at all representatives of P.

Pf: F homogeneous of degree d. Let $P = [a_1 : \dots : a_{n+1}] = [\lambda a_1 : \dots : \lambda a_{n+1}]$.

Suppose $F(a_1, \dots, a_{n+1}) = 0$. Then $F(\lambda a_1, \dots, \lambda a_{n+1}) = \lambda^d F(a_1, \dots, a_{n+1}) = 0$ □

Let $f \in k[x_1, \dots, x_{n+1}]$. Write $f = f_0 + \dots + f_d$ where f_i is homogeneous of degree i. Then $P \in \mathbb{P}^n$ is a zero of $f \iff$ P is a zero of f_i $\forall i$. (on HW)

Def: Let $S \subseteq k[x_1, \dots, x_{n+1}]$. $V(S) = \{P \in \mathbb{P}^n \mid P \text{ is a zero of each } f \in S\}$

$V(S)$ is called a <u>projective algebraic set</u>.

Note:

1.) If I = the ideal generated by S, then $V(I) = V(S)$.

2.) If $I = (f_1, \dots, f_r)$ and $f_i = \sum \underbrace{f_{ij}}_{\substack{\uparrow \\ \text{form of deg } j}}$, then $V(I) = V(\{f_{ij}\}_{i,j})$

So $V(S)$ is the set of zeros of a finite # of forms.

Def: Let $X \subseteq \mathbb{P}^n$. The <u>ideal of X</u> is

$$I(X) = \left\{ f \in k[x_1, \dots, x_{n+1}] \;\middle|\; \begin{matrix}\text{every } P \in X \text{ is} \\ \text{a zero of } f\end{matrix} \right\}$$

An ideal $I \subseteq k[x_1, \dots, x_{n+1}]$ is <u>homogeneous</u> if $\forall$ $f = \sum f_i \in I$ (f_i form of deg i), $f_i \in I$ as well.

Note: $I(X)$ is homogeneous.

Ex: $(x + y^2)$ is not homogeneous.

Prop: An ideal $I \subseteq k[x_1, \dots, x_{n+1}]$ is homogeneous $\Leftrightarrow$ it's generated by a (finite) set of forms.

Pf: Let $I = (f_1, \dots, f_r)$ and $f_i = \sum \underset{\substack{\uparrow \\ \text{form of deg } j.}}{f_{ij}}$ and suppose I is homogeneous.

Then $f_{ij} \in I \quad \forall\, i,j$ and $I \subseteq (f_{ij})_{ij} \Rightarrow I = (f_{ij})_{i,j}$.

Now suppose $I = (f_1, \dots, f_r)$, each f_i a form of deg d_i.

Suppose $g = g_m + g_{m+1} + \dots + g_s \in I$, g_j a form of deg j.

We show $g_j \in I$ by induction on m, where the base case is $m = s$, i.e. g is already homogeneous.

i.e. assume true when smallest deg of $g > m$.

$g = \sum a_i f_i$. Since each f_i is a form, we can write

$$a_i f_i = (a_{i0} + a_{i1} + \dots) f_i = \underbrace{a_{i0} f_i}_{\substack{\text{form of} \\ \text{deg } d_i}} + \underbrace{a_{i1} f_i}_{\text{deg } d_i + 1} + \dots$$

so $g_m = a_{1, m-d_1} f_1 + \dots + a_{r, m-d_r} f_r \in I$.

Thus, $g - g_m \in I$. Done by induction. $\square$

Remark: Any projective algebraic set can be written $V(I)$, where I is homogeneous, and for $X \subseteq \mathbb{P}^n$, $I(X)$ is homogeneous, so we have

$$\left\{\begin{array}{l}\text{homogeneous ideals} \\ \text{in } k[x_1, \dots, x_{n+1}]\end{array}\right\} \underset{I}{\overset{V}{\rightleftarrows}} \left\{\begin{array}{l}\text{projective algebraic} \\ \text{sets in } \mathbb{P}^n\end{array}\right\}$$

(not necessarily one-to-one!) satisfying analogous conditions as in the affine case.

Ex: Points in $\mathbb{P}^2$. Let $P=[a:b:c]\in\mathbb{P}^2$. WLOG, $c=1$, so

$P=[a:b:1]$. Let $I=(az-x,\ bz-y)$.

$P\in V(I)$, and if $Q=[\alpha:\beta:\gamma]\in V(I)$, then $Q=[a\gamma:b\gamma:\gamma]$

so $\gamma\neq 0 \Rightarrow Q=P$. i.e. $V(I)=\{P\}$.

Def: An algebraic set $V\subseteq\mathbb{P}^n$ is <u>irreducible</u> if it's not the union of two smaller algebraic sets. An irreducible algebraic set in $\mathbb{P}^n$ is a <u>projective variety</u>.

Claim: $V\subseteq\mathbb{P}^n$ irreducible $\Longleftrightarrow I(V)$ is prime.

Pf: Essentially the same as the affine case.

Affine cones

Let $V\subseteq\mathbb{P}^n$ be an algebraic set.

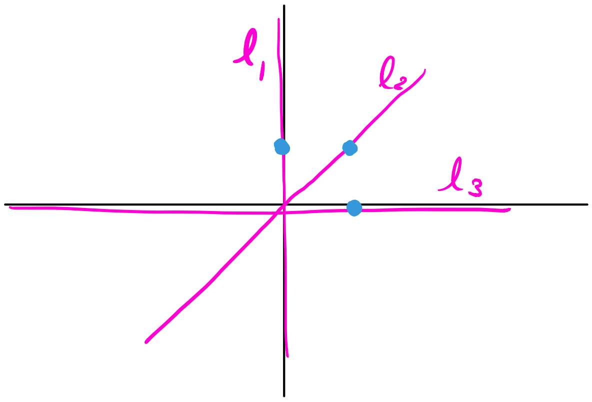
Def: The <u>affine cone</u> over V is

$$C(V)=\{(a_1,\dots,a_{n+1})\in\mathbb{A}^{n+1}\mid [a_1:\dots:a_{n+1}]\in V\}\cup\{0\}$$

i.e. the union of the corresponding lines in affine space.

Ex: 1.) $V=\{[1:0],[1:1],[0:1]\}\subseteq\mathbb{P}^1$

$C(V) =$



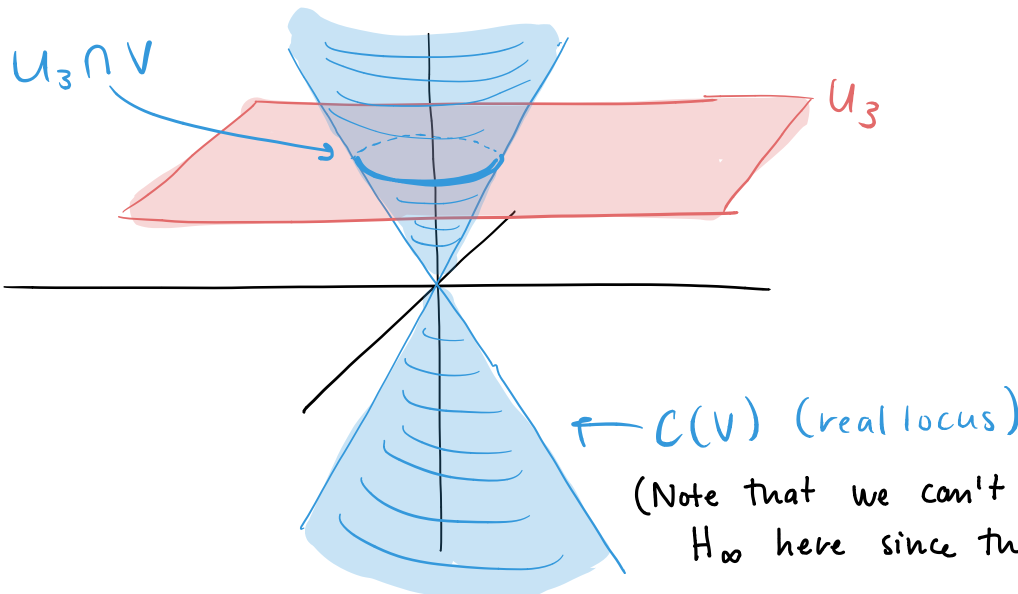
$= \ell_1 \cup \ell_2 \cup \ell_3$

2.) $I = (x^2 + y^2 - z^2) \subseteq \mathbb{C}[x,y,z]$, $\quad V = V_p(I) \subseteq \mathbb{P}^2$

to denote projective

In U_3, this is $x^2+y^2=1$. In H_∞, this is $x^2+y^2=0 \Rightarrow \underbrace{(x+iy)(x-iy)=0}_{\uparrow}$

i.e. two points in $\mathbb{P}^1$: $[i:1]$, $[-i:1]$



(Note that we can't see the two points in H_∞ here since they're not real points)

Remark: 1.) If $V \neq \emptyset$ then $I_a(C(V)) = I_p(V)$

affine — projective

2.) If I is a homogeneous ideal in $k[x_1, \dots, x_{n+1}]$ s.t. $V_p(I) \neq \emptyset$, then $C(V_p(I)) = V_a(I)$

Thm: (Projective Nullstellensatz) Let I be a homogeneous ideal in $k[x_1, \dots, x_{n+1}]$. Then

1.) $V_p(I) = \emptyset \iff$ there's an integer N s.t. I contains all forms of degree $\geq N$.

2.) If $V_p(I) \neq \emptyset$, then $I_p(V_p(I)) = \sqrt{I}$.

Pf: 1.) First we reduce to a question about affine varieties.

If $V_p(I) \neq \emptyset$, then $V_a(I) = C(V_p(I)) \supsetneq \{(0,0,\dots,0)\}$

If $V_p(I) = \emptyset$, then $V_a(I) \setminus \{0\} = \emptyset \Rightarrow V_a(I) \subseteq \{0\}$.

So we need to show $V_a(I) \subseteq \{0\} \iff (x_1, \dots, x_{n+1})^N \subseteq I$, some $N \geq 1$.

If $(x_1, \dots, x_{n+1})^N \subseteq I$ then $V_a(I) \subseteq V_a((x_1, \dots, x_{n+1})^N) = \{(0, \dots, 0)\}$.

If $V_a(I) \subseteq \{(0, \dots, 0)\}$, then $(x_1, \dots, x_{n+1}) \subseteq \sqrt{I} \Rightarrow \exists\, r > 0$ s.t. $x_i^r \in I \ \forall i$.

Let $N = r(n+1)$. Then any monomial of deg N will be divisible by x_i^r for some $i \Rightarrow (x_1, \dots, x_{n+1})^N \subseteq I$.

2.) If $V_p(I) \neq \emptyset$, then $I_p(V_p(I)) = I_a(C(V_p(I)))$

$$= I_a(V_a(I)) = \sqrt{I}. \quad \square$$

The usual corollaries hold, except we need to be careful of the <u>irrelevant ideal</u> $I_{irr} = (x_1, \dots, x_{n+1})$.

Cor: Let $S = k[x_1, \dots, x_{n+1}]$. We have the following bijective correspondences (exer)

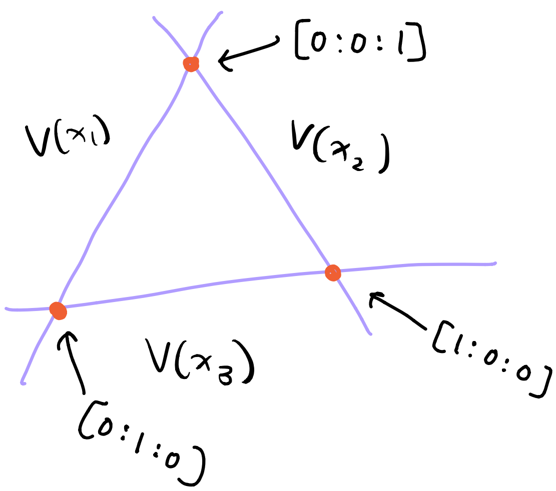
$$\left\{\begin{array}{l}\text{algebraic sets}\\ \text{in } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{homogeneous radical}\\ \text{ideals in } S\text{, other}\\ \text{than } I_{irr}\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{irreducible}\\ \text{algebraic sets}\\ \text{in } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{homogeneous prime}\\ \text{ideals, other than } I_{irr}\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{irreducible}\\ \text{hypersurfaces}\\ \text{in } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{irreducible nonconstant}\\ \text{forms, up to scaling}\end{array}\right\}$$

The hyperplanes $V(x_i)$, $i = 1, \dots, n+1$ are the <u>coordinate hyperplanes</u> or the hyperplanes at ∞ w.r.t. each U_i.

Ex: In $\mathbb{P}^2$, the $V(x_i)$ are the three <u>coordinate axes</u>.



Each pair intersects in one point.

Def: $V \subseteq \mathbb{P}^n$ a projective algebraic set is <u>Zariski closed</u>.

$\mathbb{P}^n \setminus V$ is <u>Zariski open</u>. This gives the <u>Zariski topology</u> on $\mathbb{P}^n$.

(exer: check that the subspace topology on U_i is the Zariski topology we gave to affine space.)