Bézout's Theorem: Let F and G be projective plane curves of degrees m and n respectively. Assume F and G have no common components. Then, counting multiplicity, F and G intersect in exactly mn points. That is,

$$\sum_{P \in F \cap G} I_P(F,G) = mn.$$

Pf sketch: $F \cap G$ is finite, so, after a projective change of coordinates, we can assume none of the points lie on $z=0$.

Then if f and g are the dehomogenizations w/ respect to z, we get

$$\sum_{P \in F \cap G} I_P(F,G) = \sum_P (f,g) = \sum_P \dim_k \left(\mathcal{O}_P(\mathbb{A}^2) / (f,g) \right)$$

Claim: If $I \in k[x,y]$ is an ideal and $V_a(I) = \{P_1, \dots, P_N\}$ is finite, then $k[x,y]/I \cong \prod_{i=1}^{N} \mathcal{O}_P(\mathbb{A}^2)/I$.

(For pf of claim, see Fulton section 2.9 — this is an application of the Nullstellensatz)

Thus, since $V(f,g)$ is finite, $\prod_{P \in f \cap g} \mathcal{O}_P(\mathbb{A}^2)/(f,g) \cong k[x,y]/(f,g)$.

In particular, $\sum I_P(F,G) = \dim_k \left(k[x,y]/(f,g) \right)$.

Let $\gamma = k[x,y]/(f,g)$ and $\Gamma = k[x,y,z]/(F,G)$, $R = k[x,y,z]$.

Let Γ_d and R_d be the corresponding forms of degree d.

<u>WTS</u>: $\dim \gamma = \dim \Gamma_d = mn$ for $d >> 0$.

We'll show this equality. For the first equality, see proof in Fulton.

Specifically, we'll show $\dim \Gamma_d = mn$ for $d \geq m+n$.

Consider the following maps:

$$0 \longrightarrow R \xrightarrow{\psi} R \times R \xrightarrow{\varphi} R \xrightarrow{\pi} \Gamma \longrightarrow 0$$

$\psi: c \longmapsto (Gc, -Fc)$

$\varphi: (A, B) \longmapsto AF + BG$

π: quotient map

This sequence is <u>exact</u>; i.e. the image of one map is the kernel of the next:

- $\ker \psi = 0$

- $(Gc)F + (-Fc)G = 0$, so $\operatorname{Im} \psi \subseteq \ker \varphi$

- $(A, B) \in \ker \varphi \Longrightarrow AF = -BG$.

  F and G are relatively prime, so $G \mid A$ and $F \mid B$

  $\Longrightarrow \underbrace{\alpha G}_{A} F = -\underbrace{\beta F}_{B} G \Longrightarrow \alpha = -\beta \Longrightarrow \psi(\alpha) = (\alpha G, -\alpha F)$
  $= (A, B)$.

  Thus $\operatorname{Im} \psi = \ker \varphi$.

- $\operatorname{Im} \varphi = \underbrace{(F, G)}_{\text{Ideal}} = \ker \pi$

If we restrict to forms of fixed degrees, we get another exact sequence:

$$0 \to R_{d-m-n} \to R_{d-m} \times R_{d-n} \to R_d \to \Gamma_d \to 0$$

For $d \geq 0$, we have $\dim R_d = \frac{(d+1)(d+2)}{2}$, so as long as $d \geq m+n$, we have

$$\dim \Gamma_d = \frac{(d+1)(d+2)}{2} - \frac{(d-m+1)(d-m+2)}{2} - \frac{(d-n+1)(d-n+2)}{2} + \frac{(d-m-n+1)(d-m-n+2)}{2}$$

$$= \frac{1}{2}(2mn) = mn.$$

Idea for $\dim \gamma = \dim \Gamma_d$: Show that for $d \gg 0$, a basis for Γ_d will dehomogenize to form a basis for γ. $\square$

Cor: $\sum_P m_P(F) m_P(G) \leq \deg(F) \cdot \deg(G)$

Cor: If F and G meet in exactly $\deg(F) \cdot \deg(G)$ distinct points, then they are all simple points on F and G, and all the intersections are transverse.

Cor: If two curves of deg m and n intersect in $> mn$ points, then they share a common component.

Cor: If F is a nonsingular projective plane curve, it's irreducible.

Pf: Assume F is reducible. Then $F = GH$. Then $G \cap H \neq \emptyset$, so take P in the intersection. WLOG, $P = [0:0:1]$, so dehomog., g and h have no constant term, so $m_P(F) = m_P(gh) \geq 2$. $\square$

Note that this does not hold for affine curves: $F = x(x-1)$ is reducible and nonsingular.

Ex: Singularities of irreducible curves of low degree.

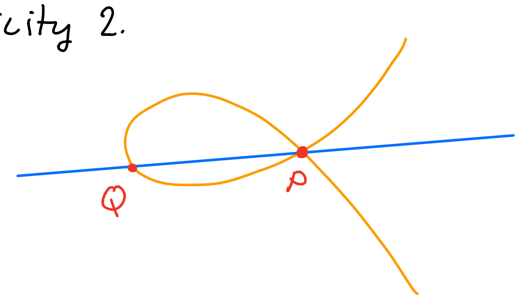
Let F be an irreducible curve of degree d.

- If $d=1$ or 2, F is nonsingular.

- If $d=3$, and P is a singular point, then let Q be any other point on F. L = line through P and Q.

Then $I_P(F,L) \geq 2$, and $I_Q(F,L) \geq 1 \Rightarrow I_P(F,L) = 2$, $I_Q(F,L) = 1$.

That is, F can have at most one singularity, and it must have multiplicity 2.



- If $d=4$, how many singular points can F have?

Trick: Let $P_1, \ldots, P_5$ be any 5 points of F.

Let V = conics through $P_1, \dots, P_5$. From last time,

$$\dim V \geq \frac{2(2+3)}{2} - 5 = 0.$$

That is, $\exists$ some conic C through $P_1, \dots, P_5$.

C and F intersect in $4 \cdot 2 = 8$ points, so $8 \geq \sum m_{P_i}(F)$.

Thus, since $2+2+2+1+1 = 8$, at most 3 pts are singular.

If 3 pts are singular, they all have mult. 2, and can't be collinear. If 2 are singular, they both have mult. 2.

If 1 point is singular, it has multiplicity at most 3.

More generally, how many singular points can a curve of degree d have?

Suppose F is an irreducible curve of degree d w/ singular pts $P_1, \dots, P_n$ of mult. $m_1, \dots, m_n$.

On the next HW, you'll prove $d(d-1) \geq \sum m_i(m_i - 1)$ just using Bézout. This is not a sharp bound though!

e.g. if $d=2$, then we get $2 \geq \sum m_i(m_i-1) = 0$

But by using what we know about linear systems, we can do better!

From last time:

$$\dim V_{d-1}\left((m_1-1)P_1,\dots,(m_n-1)P_n\right) \geq \frac{(d-1)(d-1+3)}{2} - \sum\frac{(m_i-1+1)(m_i-1)}{2}$$

$$= \underbrace{\frac{(d-1)(d+2)}{2} - \sum\frac{m_i(m_i-1)}{2}}_{\text{Set } \overset{!!}{r}} > 0$$

by above inequality

So, we can choose points $Q_1,\dots,Q_r \in F$, and get

$$\dim V_{d-1}\left((m_1-1)P_1,\dots,(m_n-1)P_n, Q_1,\dots,Q_r\right) \geq 0,$$

so $\exists$ at least one curve G in that linear system. Since F is irreducible, and $\deg G < \deg F$, the two curves can't have a common component. Thus, Bézout tells us:

$$d(d-1) \geq \sum m_i(m_i-1) + r = \sum m_i(m_i-1) - \frac{\sum m_i(m_i-1)}{2} + \frac{(d-1)(d+2)}{2}$$

$$\Rightarrow (d-1)\left(d-\frac{(d+2)}{2}\right) \geq \sum\frac{m_i(m_i-1)}{2}$$

$$\Rightarrow \boxed{(d-1)(d-2) \geq \sum m_i(m_i-1)}$$

Ex: Let $F = x^d + y^{d-1}z$. This is irreducible. The only singular point is $[0:0:1]$, and it has mult. $d-1$.

So in this case, $\sum m_i(m_i-1) = (d-1)(d-2)$, so the above bound is sharp.