Dimension

Intuitively, we "know" that $\mathbb{A}^n$ and $\mathbb{P}^n$ have dimension n. Though we already have an algebraic definition of the dimension of an affine variety, we can also give an (equivalent) definition in terms of the fields of rational functions. Roughly, it measures how many variables you need to adjoin to k to get $k(X)$, for a variety X.

Note: For $U \subseteq X$ open, $k(U) = k(X)$, so any definition of dimension involving only $k(X)$ will imply that open sets have the same dimension as the variety that they sit inside.
e.g. $\mathbb{A}^n \cong U_i \overset{\sim}{=} \mathbb{P}^n$.

Recall: If $L \subseteq K$ are fields, $L(v_1, \dots, v_n)$ is the field of fractions of $L[v_1, \dots, v_n]$ (also the smallest field containing $L, v_1, \dots, v_n$).

K is a finitely generated field extension of L if $K = L(v_1, \dots, v_n)$ for some $v_1, \dots, v_n \in K$.

Def: Let K be a f.g. field extension of k. The transcendence degree of K over k, written $\mathrm{tr.deg}_k K$ is the smallest n s.t. $\exists\, x_1, \dots, x_n \in K$ s.t. K is algebraic over $k(x_1, \dots, x_n)$.

We then say K is an algebraic function field in n variables over k.

Ex: $K = \mathbb{Q}(\sqrt{5}, \pi, x)$ has tr. deg 2 over $\mathbb{Q}$ since K is algebraic over $\mathbb{Q}(\pi, x)$

Ex: If $V = V_a(x^2 - y)$, $k = \mathbb{C}$, $K = k(V)$, then K is algebraic over $\mathbb{C}(y)$ since $x^2 - y = 0$.

Thus, K has tr. deg 1 over $\mathbb{C}$.

Def: If X is a variety, the <u>dimension</u> of X is

$$\dim(X) := \text{tr.deg}_k\, k(X).$$

Prop: Let K be an alg. function field in one variable over k. (e.g. K field of rat'l functions of a curve)

Let $x \in K$, $x \notin k$, k alg. closed.

1.) K is algebraic over $k(x)$.

2.) If $\text{char}(k) = 0$, $\exists\ y \in K$ s.t. $K = k(x, y)$. (Primitive element Thm)

3.) If R is an integral domain w/ field of fractions $= K$, s.t. $k \subseteq R$, then if $P \subsetneq R$ is a nonzero prime ideal, the map $k \longrightarrow R/P$ is an isomorphism.

Pf: 1.) Let $t \in K$ s.t. K is algebraic over $k(t)$. Then x is algebraic over $k(t)$, so $\exists$ a polynomial f s.t. $f(t, x) = 0$

x is not algebraic $/ k$ so t must appear in f.

$\Rightarrow t$ is algebraic over $k(x) \Rightarrow k(x,t)$ algebraic over $k(x)$

$\Rightarrow K$ algebraic over $k(x)$.

3.) Suppose $x \in R$ s.t. $\bar{x} \in R/p$ is not in k. Let $0 \neq y \in P$.

Choose polynomials a_i s.t. $f(x,y) = \sum a_i(x) y^i = 0$

Factoring out powers of y, we can assume $a_0(x) \neq 0$.

But then $a_0(x)$ is divisible by y, so it's in P, so $a_0(\bar{x}) = 0$

But $\bar{x}$ is not algebraic over k, so we get a contradiction. $\square$

We can now easily conclude some basic facts about dimension.

Properties of dimension:

Let X be a variety.

1. If $\emptyset \neq U \subseteq X$ is open, then $\dim U = \dim X$. $(k(U) = k(X))$
   In particular, if X is an affine variety and $\overline{X}$ its projective closure, then $\dim(X) = \dim(\overline{X})$.

2. $\dim X = 0 \iff X$ is a point.
   (Pf: If $\dim X = 0$, intersect w/ an $\mathbb{A}^n$ and take its closure to get an affine variety V. Then $k(V) = k$ since it's algebraic, so $\Gamma(V) = k. \Rightarrow V =$ a point.)

(3.) Every proper closed subvariety of a curve C is a point.
(Pf: Assume WLOG C is affine. Let $V \subseteq C$ a closed subvariety. $R = \Gamma(C)$, $P = I(V)$. Then $\Gamma(V) = R/P$, so $\Gamma(V) = k \Rightarrow k(V) = k$.)

(4.) A closed subvariety of $\mathbb{A}^2$ (resp. $\mathbb{P}^2$) has dimension one iff it's an affine (resp. projective) plane curve. (Pf: Exer.)

Rational maps

Let X, Y be varieties.

Def: A rational map f from X to Y, denoted

$$f : X \dashrightarrow Y$$

is a morphism from an open (nonempty) subvariety $U \subseteq X$ to Y, $U \to Y$ that cannot be extended to a morphism from any larger open subvariety to Y.

Ex: $f : \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ defined $x \mapsto 1/x$ is a morphism on $\mathbb{A}^1 - \{0\}$, but cannot be extended to $\mathbb{A}^1$.

In fact, any morphism from an open set $U \subseteq X$ to Y determines a unique rational map $X \dashrightarrow Y$. This is because of the following:

Claim: If $f,g : X \to Y$ are morphisms of varieties and they agree on a dense set of X then $f=g$.

Pf sketch: We can define a morphism $(f,g): X \to Y \times Y$.
Claim: the set $\Delta_Y = \{(y,y)\} \subseteq Y \times Y$ is closed.

Thus, $(f,g)^{-1}(\Delta_Y) = \{x \mid f(x)=g(x)\}$ is closed, but it contains a dense open set, so it's all of X, so $f=g$. $\square$

Since any open set is dense in X, this implies that any rational map is uniquely determined by its restriction to any open set.

Def: $f: X \dashrightarrow Y$ <u>dominant</u> if $f(U)$ is dense in Y.

Just as in the case of affine morphisms, as long as f is dominant, it induces a map on the fields of rational functions:

Prop: Let $f: X \dashrightarrow Y$ be dominant. Let $U \subseteq X$ and $V \subseteq Y$ be affine open sets s.t. $f: U \to V$ a morphism. Then

$$f^*: \Gamma(V) \to \Gamma(U)$$

is injective, so it extends to an injective map $\underset{k(Y)}{\underset{\shortparallel}{k(V)}} \to \underset{k(X)}{\underset{\shortparallel}{k(U)}}$.

Pf: Similar idea to affine case. $\square$

Q: What does f look like if it induces an isomorphism on rational functions?

Def: 1.) $f: X \dashrightarrow Y$ is <u>birational</u> if $\exists\ U \subseteq X,\ V \subseteq Y$ open s.t. $f: U \to V$ is an isomorphism.

2.) Varieties X and Y are <u>birationally equivalent</u> if $\exists$ a birational map $f: X \dashrightarrow Y$.

Ex: The map $f: \mathbb{A}^n \to \mathbb{P}^n$, $(x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_{n+1}]$ is birational.

Ex: Consider the map $f: \mathbb{P}^1 \to \mathbb{P}^2$ defined

$$[a:b] \mapsto [a^3 : ab^2 : b^3]$$

The image is $V(y^3 - xz^2)$, which is singular, so the map $f: \mathbb{P}^1 \to V(y^3 - xz^2)$ is <u>not</u> an isomorphism.

However, we'll soon see that it is birational. (In fact, the only cubics that are birationally equivalent to $\mathbb{P}^1$ are singular. The smooth cubics are elliptic curves, which are not birational to $\mathbb{P}^1$.)

Right away, we can deduce that $f: X \dashrightarrow Y$ birational
$\Rightarrow$ The induced map $f: k(Y) \to k(X)$ is an isomorphism.

In fact, the converse holds! In order to prove it, we need a little more algebra.

Def: If A and B are local rings, $A \subseteq B$, then B dominates A if $m_A \subseteq m_B$.

Lemma: Let $f: X \dashrightarrow Y$ be dominant.

1.) If $P \in X$ in the domain of f and $f(P) = Q$, then $\mathcal{O}_P(X)$ dominates $f^*(\mathcal{O}_Q(Y))$.

2.) If $P \in X$, $Q \in Y$, and $\mathcal{O}_P(X)$ dominates $f^*(\mathcal{O}_Q(Y))$, then P is in the domain of f and $f(P) = Q$.

Pf: 1.) If $\frac{a}{b} \in \mathcal{O}_Q(Y)$, then $f^*\left(\frac{a}{b}\right) = \frac{a \circ f}{b \circ f}$. $(b \circ f)(P) = b(Q) \neq 0$. Thus $f^*(\mathcal{O}_Q(Y)) \subseteq \mathcal{O}_P(Y)$. Similarly, if $\frac{a}{b} \in m_Q$, then $a(Q) = 0 \Rightarrow a \circ f(P) = 0$, so the max'l ideal maps into the max'l ideal.

2.) Take affine neighborhoods V of P, W of Q. Let

$\Gamma(W) = k[y_1, \dots, y_n]/I$. Then $f^*(y_i) = \frac{a_i}{b_i}$, $a_i, b_i \in \Gamma(V)$, $b_i(P) \neq 0$.

Then setting $b = b_1 \cdots b_n$, we have $f^*(\Gamma(W)) \subseteq \Gamma(V_b)$. But V_b is an affine variety, so this corresponds to a unique morphism $g: V_b \to W$, which thus agrees w/ f.

If $\alpha \in \Gamma(W)$ vanishes at Q, then $f^*(\alpha) \subseteq m_P$, so α vanishes at P. Thus, $g(P) = Q$. $\square$

Using the above lemma, we can show that any map between fields of rational functions is induced by a dominant rational map:

Thm: Let X and Y be varieties. Any (nonzero) homomorphism $\varphi: k(Y) \longrightarrow k(Y)$ is induced by a unique dominant rational map $X \dashrightarrow Y$.

Pf: Since any rational map on an open set of X uniquely determines a rational map on X, we can replace X and Y with open affines. Thus, assume X and Y are affine.

Then consider $\varphi(\Gamma(Y)) \subseteq k(X)$. Just as before, we can find some $b \in \Gamma(X)$, so $\varphi(\Gamma(Y)) \subseteq \Gamma(X_b)$, so we get a morphism $X_b \longrightarrow Y$ which determines a unique rational map $X \dashrightarrow Y$.

Since φ is injective, the rational map is dominant. $\square$

Now we can prove our main theorem:

Thm: X and Y are birationally equivalent $\iff k(X) \cong k(Y)$.

Pf: $\Rightarrow$: $k(X) = k(U) \cong k(V) = k(Y)$ for $U \subseteq X$ open affine, $V \subseteq Y$ open affine s.t. $U \xrightarrow{\cong} V$.

$\Leftarrow$: If $\varphi : k(X) \to k(Y)$ is an isomorphism, again replace X and Y w/ open affines.

Then, just as above, $\varphi(\Gamma(X)) \subseteq \Gamma(Y_b)$, some $b \in \Gamma(Y)$, and $\varphi^{-1}(\Gamma(Y)) \subseteq \Gamma(X_d)$, some $d \in \Gamma(X)$.

Thus, $\varphi(\Gamma((X_d)_{\varphi^{-1}(b)})) \subseteq \Gamma((Y_b)_{\varphi(d)})$ since

$$\frac{a}{d\,\varphi^{-1}(b)} \mapsto \frac{\varphi(a)}{\varphi(d)\,b},$$ and $\varphi(a) \in \Gamma(Y_b)$.

Similarly, the other inclusion holds. Thus, the corresponding spaces are isomorphic. $\square$

Def: A variety is <u>rational</u> if it is birationally equivalent to $\mathbb{P}^n$ for some n.

Open question: Are cubic fourfolds (degree 3 hypersurfaces in $\mathbb{P}^5$) all rational?

The above theorem leads to the following powerful corollary:

Corollary: Every curve is birationally equivalent to a plane curve.

Pf: For char 0: If V is a curve, $k(V)$ has tr. deg $= 1$ over k, so, from before, $k(V) = k(a,b)$.

Consider the ring map $\underset{\substack{\uparrow \\ \text{poly. ring}}}{k[x,y]} \twoheadrightarrow k[a,b] \subseteq k(V)$.

Call its kernel I. $k[a,b]$ has no zero divisors, so I is prime. Thus $V' = V(I) \subseteq \mathbb{A}^2$ is an (affine) variety.

$\Gamma(V') = k[x,y]/I \cong k[a,b]$, so $k(V') \cong k(V)$. Thus, $\dim V' = 1$ and V' is a plane curve. $\square$