Math 221, Fall 2024

Problem set 1
Due Thursday, September 12

Problem 1. Prove that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$$
$$\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$$

† (generalize this formula for the sum of an arbitrary set of ideals)

$$\sqrt{\sqrt{I}} = \sqrt{I}$$

Problem 2. Prove that prime ideals of a direct product of rings $\prod_{i=1}^{s} R_s$ are just the ideals of the form $R_1 \times R_2 \times \ldots \times \mathfrak{p}_i \times \ldots \times R_s$, where $\mathfrak{p}_i$ is a prime ideal of R_i. Prove the analogous result for maximal ideals.

Problem 3. Let $I \subset R$ be a proper radical ideal of a Noetherian ring, $V_{\min}(I) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m\}$. Prove that $\bigcup_{i=1}^{m} \mathfrak{p}_i$ is the set of *zero divisors modulo* I, that is such elements $a \in R$ that there is $b \in R \setminus I$ such that $ab \in I$. In particular, if R is Noetherian and reduced (that is without nonzero nilpotent elements), $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ are minimal prime ideals of R, then $\bigcup_{i=1}^{m} \mathfrak{p}_i$ is the set of zero divisors of R.

Definition. An element $a \in R$ is called *prime* if the ideal (a) is prime and *irreducible* if it cannot be presented as a product $a = bc$, where both b and c are non-invertible. A ring R is called *factorial* (or *unique factorization domain*, UFD) if it is a domain and every element of R can be presented as a product of prime elements.

Problem 4. Let R be a domain, $a = p_1 p_2 \ldots p_n = q_1 q_2 \ldots q_m$, where all p_i and q_j are prime. Prove that $m = n$ and there is a permutation σ of indices such that $p_i = \varepsilon_i q_{\sigma i}$ $(i = 1, 2, \ldots, n)$, where all ε_i are invertible. (Then they say that p_i and $q_{\sigma i}$ *are associated*).

Problem 5. Let R be a domain.

(1) Prove that R is factorial if and only if there are no infinite ascending chains of principal ideals

$$(a_1) \subset (a_2) \subset \ldots \subset (a_n) \subset \ldots$$

and every irreducible element of R is prime. In particular, a Noetherian domain is factorial if and only if every irreducible element is prime.

(2) Prove that if R is factorial and $a \in R$, there are finitely many minimal elements in $V(a)$ and all of them are principal ideals.

† **Problem 6.** Let X be a compact topological space (for intance, $X = [0, 1]$), $C(X)$ be the ring of continuous functions on X, $a \in X$, $\mathfrak{m}_a = \{f \in C(X) \mid f(a) = 0\}$. Prove that $\mathfrak{m}_a$ is a maximal ideal of $C(X)$ and every maximal ideal is if this form.

Problems marked with † are optional.