

Problem set 10  
Due Thursday, November 14

**Problem 1.** (1) Let  $A$  be a domain with the field of fractions  $K$ ,  $\Gamma$  be a totally ordered group and  $v_0 : A \setminus \{0\} \rightarrow \Gamma$  is such that

$$(*) \quad v_0(ab) = v_0(a) + v_0(b) \quad \text{and} \quad v_0(a + b) \geq \min\{v_0(a), v_0(b)\}.$$

Prove that there is a unique valuation  $v : K^\times \rightarrow \Gamma$  such that  $v_0 = v|_A$ .

(2) Let  $\Gamma$  be a totally ordered abelian group,  $\mathbb{k}$  be a field. Consider the ring  $\mathbb{k}\Gamma_+$  which is a vector space over  $\mathbb{k}$  with a basis  $\{[\gamma] \mid \gamma \in \Gamma, \gamma \geq 0\}$  and multiplication defined by  $\mathbb{k}$ -linearity and the rule  $[\gamma][\gamma'] = [\gamma + \gamma']$ .

(a) Prove that  $\mathbb{k}\Gamma_+$  is a domain.

(b) Construct a surjective map  $\mathbb{k}\Gamma_+ \rightarrow \Gamma$  satisfying conditions (\*).

It gives an example of valuation  $v : K^\times \rightarrow \Gamma$  with  $\text{Im } v = \Gamma$ .

**Problem 2.** A *Bezout ring* is a domain such that every finitely generated ideal in it is principal.

Prove that a local ring is a Bezout ring if and only if it is a valuation ring.

**Problem 3.** Let  $V \subset \mathbb{k}(x_1, x_2, \dots, x_n)$ , where  $\mathbb{k}$  is a field, consists of the quotients  $f/g$  with  $\deg f \leq \deg g$ . Prove that it is a discrete valuation ring with the residue field isomorphic to  $\mathbb{k}(x_1, x_2, \dots, x_{n-1})$ .

**Problem 4.** Let  $v : K^\times \rightarrow \Gamma$  be a valuation,  $V$  be the corresponding valuation ring. Prove that the following conditions are equivalent::

- (1)  $V$  is Noetherian.
- (2)  $V$  is a discrete valuation ring.
- (3)  $\text{Im } v \simeq \mathbb{Z}$ .

**Problem 5.** (1) Let  $V \subset V'$  be valuation rings with the same field of fractions  $K$  and the maximal ideals, respectively,  $\mathfrak{m}$  and  $\mathfrak{m}'$ . Prove that  $\mathfrak{m} \not\subset \mathfrak{m}'$ .

(2) Let  $A$  is a domain with the field of fractions  $K$ . Prove that there are no subrings  $A \subset B \subset K$  if and only if  $A$  is a valuation ring and  $\dim A = 1$ .