Math 221, Fall 2023

Problem set 2
Due Thursday, September 21

Problem 1. Let M be a Noetherian module, $\alpha : M \to M$ be an epimorphism. Prove that it is an isomorphism.

Problem 2. Let I be a radical ideal of a Noetherian ring A. Prove that $\bigcup_{\mathfrak{p}\in V_{\min}(I)} \mathfrak{p}$ is the set of *zero divisors modulo* I, that is elements $a \in A$ such that $ab \in I$ for some $b \notin I$. In particular, if A has no nilpotent elements, the set of zero divisors from A coincides with the union of all minimal prime ideals of A.

Problem 3. Prove that a factorial domain is integrally closed.

Problem 4. Let A be an integrally closed domain, F be its field of fractions and $f(x) \in A[x]$ be a monic polynomial. Prove that if $f(x) = g(x)h(x)$, where $g(x), h(x)$ are monic polynomials from $F[x]$, both $g(x) \in A[x]$ and $h(x) \in A[x]$.

Hint: Use the fact that there is an algebraically closed field $\tilde{F} \supseteq F$.

Problem 5. Let A be a Noetherian ring, B be an A-algebra. Prove that

1. If B is integral over A, also $B[x]$ is integral over $A[x]$.
2. If A is integrally closed in B, also $A[x]$ is integrally closed in $B[x]$.
   Hint: If $f \in B[x]$ is integral over $A[x]$, $M = A[x][f]$ is a finite $A[x]$-module. Prove that the A-submodule $\tilde{M} \subseteq B$ consisting of all coefficients of polynomials from M is finite and if b is the leading coefficient of f, then $A[b] \subseteq \tilde{M}$. Then use induction by $\deg f$.
3. Deduce that if $\mathrm{Int}(A[x], B[x]) = \mathrm{Int}(A, B)[x]$.

† **Problem 6.** Let A be a factorial domain, F be its field of fractions. A polynomial $f \in A[x]$ is called *primitive* if every common divisor of its coefficients is invertible. Prove that

1. If $f, g \in A[x]$ are primitive, so is fg.
2. Every polynomial $f \in F[x]$ can be written as $c\tilde{f}$, where $\tilde{f} \in A[x]$ is primitive and $c \in F$.
3. If $f \in A[x]$ is primitive, $f = gh$ for some $g, h \in F[x]$, then $f = \tilde{g}\tilde{h}$, where $\tilde{g}, \tilde{h} \in A[x]$ are primitive such that $g = c\tilde{g}$ and $h = c'\tilde{h}$ with $c, c' \in F$.
4. Every irreducible element from $A[x]$ is either a prime element of A or a primitive element from $A[x]$ which is prime in $F[x]$.
5. Deduce that $A[x]$ is factorial.

Problems marked with † are optional.