Algebraic Groups Homework 2

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February 14, 2023

Exercise 0.0.1 (Exercise II.2.1.9). *Using the exercise from the previous homework, compute* $\mathrm{T}_e\mathrm{SL}_n$.

Exercise 0.0.2 (Exercise II.3.4.1). *Let* k *be a field of characteristic not two and let* ω *be a symplectic form over a field* k*, then prove that*

$$\mathrm{Sp}(\mathrm{V},\omega)(\mathrm{A}) \to \mathrm{Sp}(\mathrm{V},\omega)(\mathrm{A}/\mathrm{I})$$

is surjective for any k*-algebra* A *and an ideal* I *such that* $\mathrm{I}^2 = 0$. *Describe an isomorphism between the tangent space of* $\mathrm{Sp}(\mathrm{V},\omega)$ *and* $(\mathrm{Sym}^2\,\mathrm{V})^\vee$.

Exercise 0.0.3 (Exercise II.3.4.2). *Let* k *be a field of characteristic not two; assume the classification of symmetric bilinear forms over an algebraically closed field (they are all equivalent). Prove that the* $\mathrm{O}(\mathrm{V},q)$*'s are all smooth. Conclude that* $\mathrm{SO}(\mathrm{V},q)$*'s are also smooth and describe their tangent spaces.*

Exercise 0.0.4 (Exercise III.1.2.2). *Let* A *be an abelian group and let* A^{ds} *be the category whose objects are elements of* A *and there is no non-trivial morphisms. The category of* ***A-graded vector spaces*** *is the category* $\mathrm{Fun}(\mathrm{A}^{\mathrm{ds}},\mathrm{Vect}_k)$.

(1) The group algebra of A *is the algebra* $k[\mathrm{A}]$ *whose elements are of the form*

$$\sum_{a\in\mathrm{A}} c_a a \qquad \textit{the sum is finite,}$$

and addition is given componentwise

$$\sum_{a\in\mathrm{A}} c_a a + \sum_{a'\in\mathrm{A}} b_{a'}a' := \sum_{a\in\mathrm{A}} c_a + b_a a$$

and multiplication is given by

$$\left(\sum_{a\in\mathrm{A}} c_a a\right)\left(\sum_{a'\in\mathrm{A}} b_{a'}a'\right) = \sum_{a}\sum_{a'} c_a b_{a'}(a+a').$$

Prove that $k[\mathrm{A}]$ *is naturally a Hopf algebra. Write the corresponding group scheme as* $\mathbb{G}^{\mathrm{A}}$ *(note that* $\mathbb{G}^{\mathbb{Z}} = \mathbb{G}_m$*).*

(2) Prove that we have an equivalence of categories

$$\mathrm{Fun}(\mathrm{A},\mathrm{Vect}_k) \simeq \mathrm{Rep}(\mathbb{G}^{\mathrm{A}}).$$

We note that the diagram A^{ds} *only depends on the order of* A*; what will distinguish them is the additive structure which will be reflected on the tensor structure of* $\mathrm{Fun}(\mathrm{A}^{\mathrm{ds}}, \mathrm{Vect}_k)$*, an idea that we will elaborate upon.*

Exercise 0.0.5 (Exercise III.1.2.3)**.** *We will work through the representation of* $\mathbb{G}_a$*.*

1. *observe that a representation of* $\mathbb{G}_a$ *gives rise to a map*

$$\rho : \mathrm{V} \to \mathrm{V} \otimes k[\mathrm{T}];$$

hence we write $\rho(v) = \sum_{i \geqslant 0} \rho_i(v) \otimes t^i$*. This means we have maps*

$$\rho_i : \mathrm{V} \to \mathrm{V} \qquad i \geqslant 0.$$

Use the condition on the coaction to deduce that: $\rho_0 = \mathrm{id}$ *and that*

$$\rho_i \rho_j = \binom{i+j}{i} \rho_{i+j}.$$

2. *In characteristic zero, prove that* $\rho_n = \frac{1}{n!}\rho_1$*.*
3. *Prove that, in characteristic zero, the category of* $\mathbb{G}_a$*-representation is the same thing as the category of* locally nilpotent endomorphisms*:* k*-vector spaces* V *equipped with an endomorphism* T *such that for any* $v \in \mathrm{V}$ $\mathrm{T}^{\circ k} v = 0$ *for* $k \gg 0$*.*

Exercise 0.0.6 (Exercise B.1.0.12)**.** *Check that* $\mathrm{Rep}(\mathrm{G})$ *is abelian. In other words, verify that:*

- $\mathrm{Rep}(\mathrm{G})$ *has a zero object.*
- $\mathrm{Rep}(\mathrm{G})$ *admits biproducts.*
- *Every map in* $\mathrm{Rep}(\mathrm{G})$ *has an additive inverse.*
- *Every map has a kernel and cokernel.*
- *The natural map* $\mathrm{coker}(\ker f) \to \ker(\mathrm{coker} f)$ *is an isomorphism for every map* f *in* $\mathrm{Rep}(\mathrm{G})$*.*