Algebraic Groups Homework 3

Elden Elmanto and Wyatt Reeves

March 14, 2023

Exercise 0.0.1 (Exercise III.2.5.10). *For all $0 \leqslant k \leqslant n$ we have that*

$$e(f^k v) = k(n-k+1)f^{k-1}v.$$

Prove this by induction.

Exercise 0.0.2 (Exercise 2.5.12). *Let k be a field of characteristic zero. We have the standard representation $k^{\oplus 2}$ of SL_2; we can take its symmetric algebra*

$$\mathrm{Sym}^*(k^{\times 2}) = k \oplus k^{\oplus 2} \oplus ((k^{\oplus 2})^{\otimes_2})_{\Sigma_2} \oplus \cdots ((k^{\oplus 2})^{\otimes n})_{\Sigma_n} \cdots.$$

Endow each homogeneous component with a natural SL_2-action. Prove that such the n-th component is isomorphic to the unique irreducible representation of $\mathfrak{sl}_2$ with highest weight of weight n.

We have an isomorphism

$$k[x,y] \cong \mathrm{Sym}^*(k^{\times 2})$$

and an identification of n-th homogeneous polynomial with the degree n homogeneous piece of the symmetric algebra. Compute the action of SL_2 and $\mathfrak{sl}_2$ on these homogeneous polynomials.

Exercise 0.0.3 (Exercise 2.5.13). *Let G be an algebraic group and let V be a representation of G (not assumed to be over a field of characteristic 0). By definition, $\mathrm{Sym}^2\mathrm{V}$ is $(\mathrm{V}^{\otimes 2})_{\Sigma_2}$. One can also consider $\Gamma_2\mathrm{V}$, which by definition is $(\mathrm{V}^{\otimes 2})^{\Sigma_2}$.*

(1) Show that $\Gamma_2\mathrm{V}$ can be given the structure of a subrepresentation of $\mathrm{V}^{\otimes 2}$.

(2) If V is finite-dimensional, show that $\Gamma_2(\mathrm{V}^\vee) \cong (\mathrm{Sym}^2\mathrm{V})^\vee$.

(3) In characteristic 0, show that the composition $\Gamma_2\mathrm{V} \to \mathrm{V}^{\otimes 2} \to \mathrm{Sym}^2\mathrm{V}$ is an isomorphism.

Exercise 0.0.4 (Exercise 2.5.14). *The representations of SL_2 look quite different in positive characteristic. Let's see a simple example of this. Consider SL_2/k, where k is a field of characteristic 2. Let V_{std} be the standard representation of SL_2.*

(1) Show that $\mathrm{Sym}^2\mathrm{V}_{std}$ isn't an irreducible representation of SL_2.

(2) Show that $\mathrm{Sym}^2\mathrm{V}_{std}$ isn't even semisimple.

(3) Show that $\mathrm{V}_{std} \cong \mathrm{V}_{std}^\vee$ (in any characteristic). Use Exercise 0.0.3 to conclude that $\Gamma_2\mathrm{V}_{std} \ncong \mathrm{Sym}^2\mathrm{V}_{std}$ over a field of characteristic 2.

Exercise 0.0.5 (Exercise III.3.1.5). *Prove that the above construction indeed furnishes a symmetric monoidal structure on $\mathrm{Fun}(\mathrm{A}^{\mathrm{ds}}, \mathrm{Vect}_k)$.*

Exercise 0.0.6 (Exercise III.3.2.3). *The key example is of course $\mathcal{C} = \mathrm{Vect}_k$, the category of k-vector spaces. A vector space V is dualizable if and only if it is finite dimensional. The key point here (and the only thing you need to explain) is that why the fact that the the two composites in the definition above being the identity implies that V must be finite dimensional.*