Algebraic Groups Homework 4

Elden Elmanto and Wyatt Reeves

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0.1 SL_3

The goal of this part of the homework is to understand SL_3. Let's work over a field of characteristic 0 throughout this section. We've seen already that $\mathfrak{sl}_3$ is the space of traceless matrices. Let's now specify a nice basis for $\mathfrak{sl}_3$ and fix notation:

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad f_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let's also write $h_3 = h_1 + h_2$.

Exercise 0.1.1. *Compute the Lie bracket for* SL_3 *for all pairs of the eight basis vectors above.*

Exercise 0.1.2. *Note that* $\{e_i, f_i, h_i\}$ *forms an* $\mathfrak{sl}_2$ *subalgebra of* $\mathfrak{sl}_3$ *for each* $i = 1, 2, 3$. *Determine its decomposition of the adjoint representation of* $\mathfrak{sl}_3$ *into irreducible* $\mathfrak{sl}_2$*-representations after restriction to the copies* $\mathfrak{sl}_2$ *given by* $\{e_1, f_1, h_1\}$ *and and* $\{e_2, f_2, h_2\}$.

Exercise 0.1.3. *Show that the adjoint representation of* $\mathfrak{sl}_3$ *is simple.*

Definition 0.1.4. The **trace form** on $\mathfrak{sl}_3$ is the bilinear form $\mathrm{X}, \mathrm{Y} \mapsto \mathrm{tr}(\mathrm{XY})$.

Definition 0.1.5. Note that SL_3 contains a torus T given by the diagonal matrices. This torus is isomorphic to $\mathbf{G}_m^2$, for instance by the map $\mathbf{G}_m^2 \to \mathrm{T}$ given by

$$(a, b) \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1}b & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}$$

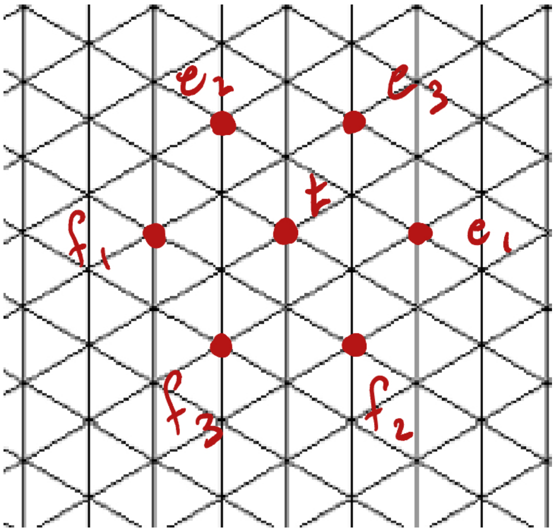
Write $\mathfrak{t}$ for the Lie algebra of T.

Exercise 0.1.6. *Show that the trace form is nondegenerate, symmetric, and invariant under the adjoint action of* SL_3. *Show that its restriction to* $\mathfrak{t}$ *is positive-definite.*

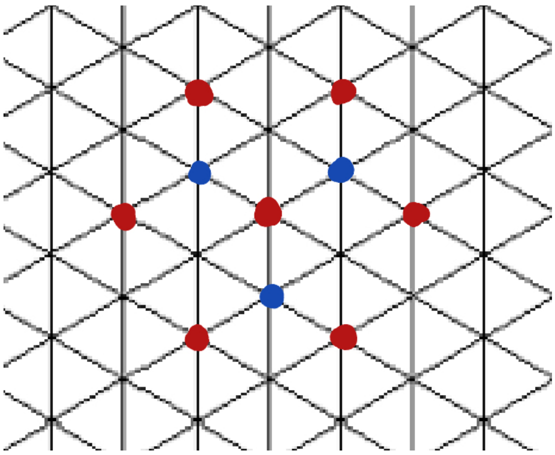
Definition 0.1.7. Given a vector space V equipped with a positive-definite symmetric bilinear form κ, the **angle** $\theta(v, w)$ between two vectors $v, w \in \mathrm{V}$ is given by the formula

$$\cos\theta(v, w) = \frac{\kappa(v, w)}{\sqrt{\kappa(v, v)\kappa(w, w)}}$$

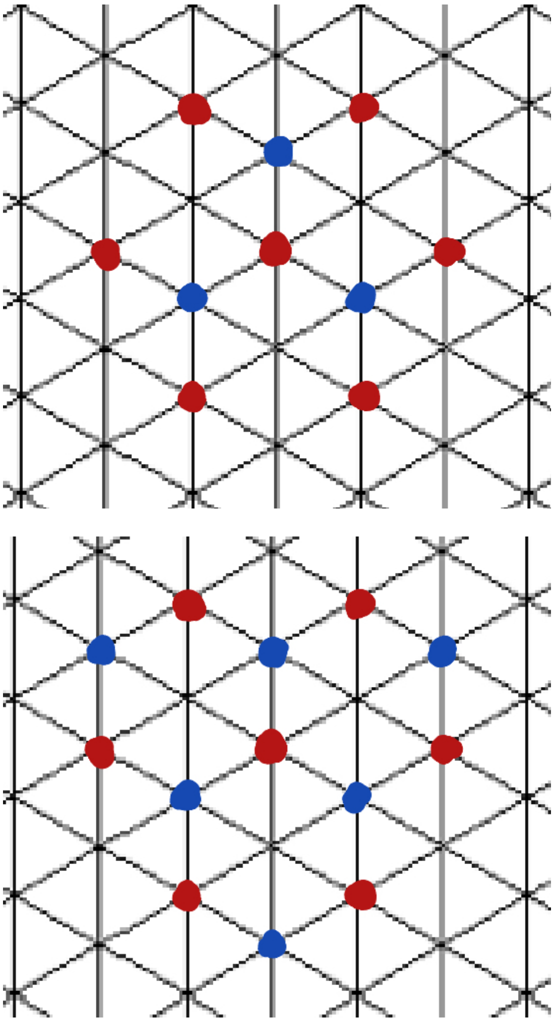
Exercise 0.1.8. *Given a weight $\lambda : \mathrm{T} \to \mathbf{G}_m$, taking differentials gives $d\lambda : \mathfrak{t} \to k$, which we can identify with an element of $\mathfrak{t}^\vee$. The trace form induces a positive-definite symmetric bilinear form on $\mathfrak{t}^\vee$. Show that the angles in the following picture agree with the angles computed using this form. (In the picture we abuse notation by identifying a weight space for T in the adjoint representation of $\mathfrak{sl}_3$ with its corresponding weight, and that weight with $d\lambda$ in $\mathfrak{t}^\vee$.)*



Exercise 0.1.9. *Show that the standard representation of SL_3 has weights given by the blue dots in the following image. (Like before, this picture is taking place in $\mathfrak{t}^\vee$. We haven't labelled the red dots to reduce clutter, but you should take them to correspond to the same weights as the previous exercise.)*



Exercise 0.1.10. *For each of the following two images, find a representation of SL_3 that has weight spaces of multiplicity one, given by the blue dots.*



0.2 PGL_2

Exercise 0.2.1. *Let $\mathbf{G}_m$ include into GL_n as scalar matrices (i.e. diagonal matrices where all the diagonal entries are equal). Let μ_n include into SL_n as $\mathrm{SL}_n \cap \mathbf{G}_m$ (where the intersection is formed inside of GL_n). Show that $\mathrm{GL}_n/\mathbf{G}_m \cong \mathrm{SL}_n/\mu_n$.*

Definition 0.2.2. PGL_n is by definition $\mathrm{GL}_n/\mathbf{G}_m$.

Definition 0.2.3. Given a map of algebraic groups $\mathrm{G} \to \mathrm{H}$ and a representation V of H, we can form the G-representation $\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}\mathrm{V}$, whose underlying vector space is V and the action of G is given by the composition $\mathrm{G} \to \mathrm{H} \to \mathrm{GL(V)}$.

Exercise 0.2.4. *Show that if $\mathrm{G} \to \mathrm{H}$ is a faithfully-flat map of algebraic groups then V is an irreducible H representation if and only if $\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}\mathrm{V}$ is an irreducible G-representation.*

Exercise 0.2.5. *By 0.2.1 and 0.2.4, every irreducible representation of PGL_2 restricts to an irreducible representation of SL_2. Use this to classify the irreducible representations of PGL_2 over a field of characteristic 0.*