Notes on algebraic groups and their representations

Elden Elmanto

Wyatt Reeves

Department of Mathematics, Harvard University, 1 Oxford St., Cambridge, MA 02138, USA
E-mail address: `elmanto@math.harvard.edu`
URL: `https://www.eldenelmanto.com/`

Contents

CHAPTER 1

Introduction

Our goal in this class is to use algebraic geometry to understand (certain) groups. Why might this be a good idea? Well, many of the most interesting groups are groups of matrices, and we learn early on that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & bf+dh \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Notably, the product and inverse operations on matrices can be specified by polynomial and rational equations. This is exactly the domain of *algebraic* geometry, and we will gain a lot of insight into the structure and representation theory of matrix groups by understanding them geometrically.

CHAPTER 2

Basics

1. Algebraic groups

Let $\mathcal{C}$ be a category with products and a terminal object denoted by $*$. Then a **group object** or, simply, a **group** in $\mathcal{C}$ is a quadruple $(\mathrm{G}, m, e, \iota)$ where $\mathrm{G} \in \mathcal{C}$ and

$$m : \mathrm{G} \times \mathrm{G} \to \mathrm{G}$$

is the “multiplication”,

$$e : * \to \mathrm{G}$$

is the “identity”, and

$$\iota : \mathrm{G} \to \mathrm{G}$$

is the inverse. These are subject to the commutativity of the following diagrams:

(Associativity)

$$\begin{tikzcd}
\mathrm{G} \times \mathrm{G} \times \mathrm{G} \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] & \mathrm{G} \times \mathrm{G} \arrow[d, "m"] \\
\mathrm{G} \times \mathrm{G} \arrow[r, "m"] & \mathrm{G}.
\end{tikzcd}$$

(Identity)

$$\begin{tikzcd}
* \times \mathrm{G} \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\simeq"'] & \mathrm{G} \times \mathrm{G} \arrow[d, "m"] & \mathrm{G} \times * \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "\simeq"] \\
 & \mathrm{G} &
\end{tikzcd}$$

(Inverses)

$$\begin{tikzcd}
\mathrm{G} \arrow[r, "\iota \times \mathrm{id}"] \arrow[d] & \mathrm{G} \times \mathrm{G} \arrow[d, "m"] & \mathrm{G} \arrow[l, "\mathrm{id} \times \iota"'] \arrow[d] \\
* \arrow[r, "e"'] & \mathrm{G} & *. \arrow[l, "e"]
\end{tikzcd}$$

Exercise 1.0.1. *Prove that there is an isomorphism* $\mathrm{X} \times * \simeq \mathrm{X}$ *for any* $\mathrm{X} \in \mathcal{C}$.

We denote by $\mathrm{Grp}(\mathcal{C})$ the category of groups in $\mathcal{C}$, the morphisms are morphisms in $\mathcal{C}$, $\varphi : \mathrm{G} \to \mathrm{G}'$ such that the following diagrams commute

$$\begin{tikzcd}
\mathrm{G} \times \mathrm{G} \arrow[r, "\varphi \times \varphi"] \arrow[d, "m"'] & \mathrm{G}' \times \mathrm{G}' \arrow[d, "m"] \\
\mathrm{G} \arrow[r, "\varphi"] & \mathrm{G}'.
\end{tikzcd}$$

We also note that we always have the forgetful functor $\mathrm{Grp}(\mathcal{C}) \to \mathcal{C}$ which extracts G.

Exercise 1.0.2. *Prove the forgetful functor* $\mathrm{Grp}(\mathcal{C}) \to \mathcal{C}$ *is conservative, i.e., a morphism* $\varphi : \mathrm{G} \to \mathrm{G}'$ *is an isomorphims if and only if it is an isomorphims as a morphism in* $\mathcal{C}$.

Example 1.0.3. A group in the category Set is the usual notion of a group.

Example 1.0.4. We have the category Mfd of manifolds. Then objects in Grp(Mfd) are called **Lie groups**.

The focus of this class will be the notion of group objects in the category of schemes, but pretty soon we will focus only on group objects in affine varieties. We begin by letting k be a ring, though very quickly it will be a field. We write Sch_k to be the category of k-schemes; we also write $* = \operatorname{Spec} k$. If k is a field, recall that a **k-variety** or, if the context is clear, a **variety**, is an integral k-scheme X such that the morphism $\mathrm{X} \to \operatorname{Spec} k$ is separated and finite type.

Definition 1.0.5. An **algebraic group over** k (or, simply, an **algebraic group** if the context is clear; we also sometimes refer to these things as **group schemes**) is a group object G in Sch_k. If G is affine, then we call it a **affine algebraic group**.

Before we go on, we make a remark on extension of scalars. Recall that if $k \to k'$ is a morphism of rings, then we have the base change functor

$$\mathrm{Sch}_k \to \mathrm{Sch}_{k'} \qquad \mathrm{X} \mapsto \mathrm{X} \times_k k' =: \mathrm{X}_{k'}$$

which, for an affine scheme $\operatorname{Spec} \mathrm{R}$, we get $(\operatorname{Spec} \mathrm{R})_{k'} = \operatorname{Spec} \mathrm{R} \otimes_k k'$.

Exercise 1.0.6. *Prove that* $\mathrm{G}_{k'}$ *is an algebraic group if* G *is.*

One of the features of the algebraic group theory is that there possible **forms** of groups, i.e., variations on how the group "looks like" as we change scalars.

2. Examples and how to verify them

2.1. Algebraic group as group-valued functors. Let X be a k-scheme. Then we can consider the following functor

$$\mathrm{X} : \mathrm{CAlg}_k \to \mathrm{Set} \qquad \mathrm{R} \mapsto \mathrm{Hom}_{\mathrm{Sch}_k}(\operatorname{Spec} \mathrm{R}, \mathrm{X}).$$

The functor above is called X as well precisely because there is no loss of information:

Lemma 2.1.1. *The functor*

$$\mathrm{Sch}_k \to \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$$

described above is fully faithful.

The point of the above lemma is a combination of Yoneda's lemma plus descent. We will not get into the proof, but it provides a way to think about algebraic groups: we have a forgetful functor $\mathrm{Grp} \to \mathrm{Set}$ and we can consider group-valued functors, i.e., those functors that factors as

$$\begin{tikzcd} & \mathrm{Grp} \arrow[d] \\ \mathrm{CAlg}_k \arrow[r] \arrow[ur, dashed] & \mathrm{Set}. \end{tikzcd}$$

Lemma 2.1.2. *We have the following commutative diagram of categories*

$$\begin{tikzcd} \mathrm{Grp}(\mathrm{Sch}_k) \arrow[r, hook] \arrow[d] & \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Grp}) \arrow[d] \\ \mathrm{Sch}_k \arrow[r, hook] & \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}). \end{tikzcd}$$

Therefore the data of a group scheme over k consists of specifying for each k-algebra R, a group $\mathrm{G}(\mathrm{R})$ and for each morphism $\mathrm{R} \to \mathrm{R}'$ a group homomorphism $\mathrm{G}(\mathrm{R}) \to \mathrm{G}(\mathrm{R}')$ subject to compatibilities such that we get a functor. This gives a very simple proof of the following:

Exercise 2.1.3. *Explain why it is immediate that a morphism of groups* $\varphi : \mathrm{G} \to \mathrm{G}'$ *preserve identities and inverses.*

Of course, a random functor from CAlg_k to Set need not come from a scheme, but in practice we will be handed an obvious scheme which we can then verify has a functor-of-points which land in the category of groups. All of this makes it convenient to actually specify algebraic groups.

Example 2.1.4 (The additive group). Recall the scheme $\mathbb{A}^1_k$ represents the forgetful functor:

$$\mathbb{A}^1_k(\mathrm{R}) = \mathrm{R}.$$

A ring is, in particular, a group. Hence we obtain the structure of an algebraic group on $\mathbb{A}^1_k$; in the literature this algebraic group is often called the **additive group**. Similarly, we obtain the structure of an algebraic group on $\mathbb{A}^n_k$.

Example 2.1.5 (The multiplicative group). Recall the scheme $\mathbb{A}^1_k \smallsetminus \{0\}$ represents invertible elements in a ring:

$$\mathbb{A}^1_k \smallsetminus \{0\}(\mathrm{R}) = \mathrm{R}^\times.$$

Hence, obtain the structure of an algebraic group on $\mathbb{A}^1_k \smallsetminus \{0\}$. We often denote this algebraic group as $(\mathbb{G}_m)_k$ and call it the **multiplicative group**.

These groups are both **commutative** which can be expressed via diagrams or the fact that the functor-of-points factor through the category of abelian groups.

Example 2.1.6 (The general and special linear groups). As a scheme, the general linear group is the complement of the divisor $\det \hookrightarrow \mathbb{A}^{n\times n}_k$. It is also the affine scheme

$$\operatorname{Spec} k[\mathrm{T}_{11}, \cdots, \mathrm{T}_{ij}, \cdots, \mathrm{T}_{nn}]\left(\det^{-1}\right);$$

we denote it by $(\mathrm{GL}_n)_k$. What are its functor-of-points? Well it is easy to see that $\mathrm{GL}_n(\mathrm{R})$ is the matrix group of invertible $n \times n$-matrices with the usual law of matrix multiplication.

For each R, we also have $\mathrm{SL}_n(\mathrm{R}) \subset \mathrm{GL}_n(\mathrm{R})$ of those matrices of determinant 1. This is also an algebraic group, defined as the closed subscheme of $(\mathrm{GL}_n)_k$ where $\det = 1$. We should then define the notion of a subgroup in this context.

Suppose that k is a field, and V is a finite dimensional vector space of rank n. We have a coordinate-free description of GL_n as the algebraic group $\mathrm{GL}(\mathrm{V})$. Its functor-of-points is given by $\mathrm{R} \mapsto \mathrm{GL}(\mathrm{V}_\mathrm{R})$ where the latter means R-linear automorphisms of V_R. Choosing a basis is equivalent to given an identification of algebraic groups $\mathrm{GL}(\mathrm{V}) \simeq \mathrm{GL}_n$.

Definition 2.1.7. If G is an algebraic group, then a **subgroup** of G is a closed subscheme $\mathrm{H} \hookrightarrow \mathrm{G}$ such that we have factorizations

$$\begin{tikzcd}
* \arrow[d, "e"] \arrow[dr, dashed] & \\
\mathrm{G} & \mathrm{H} \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{G}\times\mathrm{G} \arrow[d, "m"] & \mathrm{H}\times\mathrm{H} \arrow[l] \arrow[d, dashed] \\
\mathrm{G} & \mathrm{H}. \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{G} \arrow[d, "\iota"] & \mathrm{H} \arrow[l, hook'] \arrow[d, dashed] \\
\mathrm{G} & \mathrm{H}. \arrow[l, hook']
\end{tikzcd}$$

It is easy to verify that SL_n is a subgroup of GL_n via the following general fact since it is the kernel of the determinant homomorphism $\mathrm{GL}_n \to \mathbb{G}_m$.

Exercise 2.1.8. *The* ***kernel*** *of a morphism of algebraic groups* $\varphi : \mathrm{G} \to \mathrm{H}$ *is defined as the pullback*

$$\begin{tikzcd}
\ker(\varphi) \arrow[r] \arrow[d] & \mathrm{G} \arrow[d, "\varphi"] \\
* \arrow[r, "e"] & \mathrm{H}.
\end{tikzcd}$$

Prove that $\ker(\varphi) \hookrightarrow \mathrm{G}$ *is always a subgroup.*

Exercise 2.1.9. *Using Exercise* **??**, *compute* $\mathrm{T}_e\mathrm{SL}_n$.

Example 2.1.10. A **(split)** k**-torus** is an algebraic group T such that $\mathrm{T} \cong (\mathbb{G}_m)_k^n$ for some $n \geqslant 1$. If k is a field, then a k**-torus** is an algebraic group over k such that there exists a finite separable extension L of k one such that $\mathrm{T}_\mathrm{L} \cong (\mathbb{G}_m)_\mathrm{L}^n$. The exercise below illustrates this:

Exercise 2.1.11. *Consider the affine scheme over* $\mathbb{R}$ *which we denote by* X *defined by the equation* $x^2 + y^2 = 1$. *Prove that naturally can be given the structure of a* $\mathbb{R}$*-algebraic group (hint: after looking at the example of orthogonal groups below, prove that* $\mathrm{X} \cong \mathrm{SO}(x^2 + y^2)$*) to obtain the group structure) . Prove that this scheme is not isomorphic, over* $\mathbb{R}$*, with* $(\mathbb{G}_m)_\mathbb{R}$. *However, prove that* $\mathrm{X}_\mathbb{C}$ *is a (split)* $\mathbb{C}$*-torus. This is an example of a non-split torus.*

Example 2.1.12. Let k be a field of characteristic not two. A **nondegenerate quadratic space**, which is a finite dimensional vector space V with a nondegenerate quadratic form q[1]. The orthogonal group is the subgroup of GL(V) given by the functor of points:

$$\mathrm{R} \mapsto \{g : q(gv) = q(v) \forall v \in \mathrm{V}_\mathrm{R}\} \subset \mathrm{GL(V)}.$$

We write it as $\mathrm{O}(\mathrm{V}, q)$; these are the **orthogonal groups**. For an explicit example of a quadratic form that one might be interested in we have either the even ones:

$$x_1x_2 + x_2x_3 + \cdots x_{2n-1}x_{2n}$$

or the odd ones

$$x_0^2 + x_1x_2 + x_2x_3 + \cdots x_{2n-1}x_{2n}.$$

These things actually behave differently as we shall see. We also have $\mathrm{SO}(\mathrm{V}, q)$ the **special orthogonal groups** which are defined as the kernel of the determinant map $\mathrm{O}(\mathrm{V}, q) \to \mathbb{G}_m$.

Example 2.1.13. Let k be a field, again of characteristic not two. Recall that we have the **symplectic form**:

$$\begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0. \end{bmatrix}$$

which is an example of a nondegenerate alternating form on a vector space V (necessarily of dimension $2n$). More generally, we let ω be a nondegenerate alternating form on V. Then we can consider the subgroup of GL(V) given by functor of points

$$\mathrm{R} \mapsto \{g : \omega(gv, gw) = \omega(v, w) \forall v, w \in \mathrm{V}_\mathrm{R}\}.$$

For the standard symplectic matrix we denote the corresponding algebraic group by Sp_{2n} and, more generally, we write $\mathrm{Sp}(\mathrm{V}, \omega)$; these are the **symplectic groups**.

Exercise 2.1.14. *Prove that* $\mathrm{O}(\mathrm{V}, q)$ *and* $\mathrm{Sp}(\mathrm{V}, \omega)$ *are indeed schemes.*

Next, we consider group schemes which feel a bit different:

Definition 2.1.15. A k-algebraic group G is **finite** if the structure morphism $\mathrm{G} \to *$ is finite. In the literature, these are often called **finite group schemes**

---

[1] Recall that a **quadratic form** is a function $q : \mathrm{V} \to k$ such that $q(\alpha x) = \alpha^2 q(x)$ and such that its **polarization**:

$$(x, y) := q(x + y) - q(x) - q(y)$$

is bilinear; we say that it is **nondegenerate** if the associated bilinear form is.

This exactly means that $\mathcal{O}(\mathrm{G})$ is finite dimensional k-vector space.

Example 2.1.16. The easiest examples come from finite group theory: if G is a finite group we take $|\mathrm{G}|$ copies of $*$ and consider it as a group. This gives a functor $\mathrm{Grp} \to \mathrm{Grp}(\mathrm{Set})$ landing in finite group schemes which we again denote by G. Note that $|\mathrm{G}|$ is exactly the dimension of $\mathcal{O}(\mathrm{G})$

Example 2.1.17. We define the group of **m-th roots of unity** by $\mu_m := \ker(\mathbb{G}_m \xrightarrow{x \mapsto x^m} \mathbb{G}_m)$. This is a finite group scheme which is not of the form above (in general); it is the Spec of the ring $k[\mathrm{T}]/(\mathrm{T}^m - 1)$

Exercise 2.1.18. *Prove that* $\ker(\varphi)$ *is a subgroup of* G*; that is: it is a closed subscheme of* G *which inherits the group structure of* G*.*

Example 2.1.19. There are interesting finite group schemes whose underlying space is just a point; let k be a field which is characteristic $p > 0$, then:

1. we have μ_{p^n} as defined above;
2. we have α_{p^n} which is the kernel of the map
$$\mathbb{G}_a \xrightarrow{x \mapsto x^{p^n}} \mathbb{G}_a.$$
We note that this is well-defined as a morphism of groups only in characteristic $p > 0$. The ring of functions of α_{p^n} is given by $k[\mathrm{T}]/(\mathrm{T}^{p^n})$.

Exercise 2.1.20. *Prove that* α_{p^n} *and* μ_{p^n} *are isomorphic as* k*-schemes but not isomorphic as algebraic groups.*

2.2. Hopf algebras. Most of the groups we have seen so far are affine and we will stick to this for the most part. This will remain the case for most of the class. An example to keep in mind:

Example 2.2.1. An **elliptic curve** over a field k is a smooth, projective 1-dimensional variety of genus 1 equipped with a prescribed point $0 \in \mathrm{E}$. By definition, this cannot be affine. Away from characteristics two and three, these are defined by homogeneous equations of the form $y^2 = x^3 + ax + b$. A standard fact that is that an elliptic curve is a group variety, which is furthermore *commutative*. These kind of groups, which we will mostly ignore for the rest of class, are called **anti-affine** which are defined to be those algebraic groups such that $\mathcal{O}(\mathrm{G}) = k$.

Now, if G is affine, the ring of functions $\mathcal{O}(\mathrm{G})$, a k-algebra, completely describes G.

Definition 2.2.2. A **Hopf algebra over** k (or, simply a **Hopf algebra** if the context is clear) is a k-algebra A equipped with morphisms of k-algebras called a comultiplication
$$\Delta : \mathrm{A} \to \mathrm{A} \otimes_k \mathrm{A},$$
an augmentation
$$e : \mathrm{A} \to k,$$
and an antipode
$$\iota : \mathrm{A} \to \mathrm{A}$$
such that taking Spec gives $\operatorname{Spec} \mathrm{A}$ the structure of a k-algebraic group.

Appropriately defined the category of Hopf algebras is equivalent to the category of k-groups which are affine. These are all tedious checks. One utility of this language is to do computations with representations of G, another gives us a way to think about subgroups (the same way that ideals let us think about closed subschemes):

Definition 2.2.3. A **Hopf ideal** is an ideal I of A such that:
$$\Delta(\mathrm{I}) \subset \mathrm{A} \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{A}$$
$$\epsilon(\mathrm{I}) = 0$$
$$\iota(\mathrm{I}) \subset \mathrm{I}.$$

Lemma 2.2.4. *There is a 1-1 inclusion reversing bijection between Hopf ideals and subgroups of* $\operatorname{Spec} A$.

Proof. It suffices to note that A/I is a Hopf algebra if I is a Hopf ideal. We will leave verification to the reader.

□

Exercise 2.2.5. *Write down the comultiplication for* $\mathcal{O}(GL_n)$ *as a Hopf algebra, verify that* $(\det - 1)$ *is a Hopf ideal. Write down explicitly the antipode and the augmentation for* SL_2.

3. Schematic properties of algebraic groups

We work over a fixed field k for the rest of this chapter. We now delve into some geometric properties of algebraic groups, chief among which is that it is smooth in characteristic zero. More generally, the smoothness of a group is entirely about the absence of nilpotents as we will see later.

3.1. Separatedness. To begin with, we note that algebraic groups over k are, in general, separated schemes.

Exercise 3.1.1. *Let* X *be a* k*-scheme, prove that if* $\iota : \operatorname{Spec} k \to X$ *is a section of the structure morphism, then* ι *is a closed immersion. On the other hand, show that sections of morphisms of schemes in general need not be a closed immersion.*

Algebraic groups over k do not suffer pathologies like the "bug-eyed point":

Proposition 3.1.2. *Any algebraic group over* k *is separated.*

Proof. By the above exercise the morphism $e : * \to X$ is closed immersion. Now, the diagonal map $\Delta : G \to G \times G$ fits into the pullback diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G \times G \arrow[d, "m(\mathrm{id}\times\iota)"] \\
* \arrow[r, "e"] & G,
\end{tikzcd}$$

whence the claim follows from the fact that closed immersions are stable under pullback. □

3.2. Connectedness. In scheme theory, a connected component containing a point x refers to the a maximal connected subset of the underlying topological space. Every point is contained in a connected component and that connected components are closed but it might not be open (which is a little weird). Noetherian assumptions ensure that this does not happen: in fact locally noetherian schemes have connected components which are closed and open (hence clopen).

Remark 3.2.1. The functorial perspective should always be taken into account when trying to give some intuition about algebraic groups. Let us have a look at $\mathbb{G}_m = GL_1$. If we look at $\mathbb{G}_m(\mathbb{R})$, then we are looking at $(\mathbb{R} \setminus \{0\})$ which is, topologically, two points. This is disconnected as a topological space. On the other hand, if we look at $\mathbb{G}_m(\mathbb{C})$ we are looking at $\mathbb{C} \setminus 0$ which is, topologically, the circle S^1. This latter space is connected.

Clearly finite groups cannot ever be connected. We instead write G^0 for the connected component of the identity of G, i.e., this is the largest connected component of G containing the image of e. In particular, we have a clopen immersion $G^0 \hookrightarrow G$.

Lemma 3.2.2. G^0 *is a subgroup of* G.

Proof. We first claim that the map $G^0 \times G^0 \to G$ factors through G^0. By standard facts from topology, it suffices to prove that $G^0 \times G^0$ is connected; it is not true that the product of connected schemes is connected as explained in the remark below. What is true, however, is that G^0 is actually *geometrically connected*, in which case the claim is true.

□

REMARK 3.2.3. Let $\mathrm{F} \to \mathrm{L}$ be a finite Galois extension with Galois group G, then $\operatorname{Spec} \mathrm{L}$ is clearly a connected scheme (it is just a single point), but $\operatorname{Spec} \mathrm{L} \times_{\operatorname{Spec} \mathrm{F}} \operatorname{Spec} \mathrm{L} \cong \operatorname{Spec} \mathrm{L} \otimes_{\mathrm{F}} \mathrm{L} \cong \sqcup_{\mathrm{G}} \operatorname{Spec} \mathrm{L}$. One sees that the problem here is arithmetic in nature: $\operatorname{Spec} \mathrm{L}$ is connected but not geometrically connected.

Recall that a scheme X is said to be **geometrically connected** if for any separable algebraic closure $\overline{k}$ of k, $\mathrm{X}_{\overline{k}}$ is connected. The above remark shows that L is not geometrically connected, even though it is connected. To prove the above result, we will need the following result from algebraic geometry:

LEMMA 3.2.4. *Let* X *be a* k*-scheme. Then:*

1. *if* X *has a* k*-rational point, and* X *is connected then* X *is geometrically connected;*
2. *if* Y *is connected and* X *is geometrically connected then* $\mathrm{X} \times_k \mathrm{Y}$ *is connected.*

PROOF. These are [**?**, Tag 04KV] and [**?**, Tag 0385] respectively.

□

EXERCISE 3.2.5. *Prove that* SL_n *is connected over arbitrary fields.*

EXERCISE 3.2.6. *Let k be an algebraically closed field of characteristic not two. We remind the reader that any quadratic form (of rank n) in this case is isomorphic to* $x_1^2 + \cdots x_n^2$*. Prove that* O_n *is not connected (hint: use the determinant to construct a map* $\mathrm{O}_n \to * \sqcup *$*). Describe* $(\mathrm{O}_n)^\circ$*.*

3.2.7. *A remark on the algebraic versus analytic theory.* Consider the Lie group $(\mathbb{R}^\times, \cdot)$; clearly it is disconnected, but we can nonetheless take the neutral component $((\mathbb{R}^\times)^0, \cdot)$ which is itself a Lie group. Now we have the exponential and logarithm which defines isomorphisms of Lie groups

$$((\mathbb{R}^\times)^0, \cdot) \cong (\mathbb{R}, +).$$

In some sense, this says that in the world of Lie groups $(\mathbb{G}_m)^0 \simeq \mathbb{G}_a$. This is plain false in the world of algebraic groups and is one of the key differences of the algebraic theory from the analytic theory.

3.3. Smoothness. We now discuss smoothness, probably the first important idea in the course; for more details we refer the reader to Section **??**.

DEFINITION 3.3.1. Let k be a field and X a locally finite type k-scheme. Then X is **smooth** if $\mathrm{X}_{\overline{k}}$ is a regular scheme: for all $x \in \mathrm{X}$ the local rings $\mathcal{O}_{\mathrm{X},x}$ satisfy: $\dim(\mathcal{O}_{\mathrm{X},x}) = \dim(\mathfrak{m}/\mathfrak{m}^2)$.

REMARK 3.3.2. In fact, one of the crucial points of modern algebraic geometry is that smoothness is a relative notion, unlike regularity which is an absolute notion. In other words it is not quite appropriate to say that X is "*smooth*" but rather that the structure map $\operatorname{Spec} k$ is smooth. For the purposes of this class we will mostly only discuss what it means for a scheme to be smooth over a field.

REMARK 3.3.3. In Section **??** we discuss three perspectives on smoothness:

1. first, we have the Jacobian criterion which is a way to verify smoothness given defining equations,
2. the Cohen's structure theorem which characterizes regular local rings concrete,
3. the functorial point of view or *Grothendieck's criterion*

All three perspectives are important and have their own merits and the reader is encouraged to flip to the appendix to read about them.

Some algebraic groups are smooth in an obvious way. Let us verify this in the case of GL_n, unconditionally over any field.

LEMMA 3.3.4. *The algebraic group* GL_n *is smooth over any field.*

Proof. Let us use Grothendieck's criterion: we have a map $\mathrm{A} \to \mathrm{A}/\mathrm{I}$ of k-algebras such that $\mathrm{I}^2 = 0$. Our goal is to prove that the map

$$\mathrm{GL}_n(\mathrm{A}) \to \mathrm{GL}_n(\mathrm{A}/\mathrm{I})$$

is surjective. To even be able to do this in general let us examine $n = 1$: we are asking that $\mathbb{G}_m(\mathrm{A}) \to \mathbb{G}_m(\mathrm{A}/\mathrm{I})$ is surjective, i.e, that the map of unit groups

$$\mathrm{A}^\times \to (\mathrm{A}/\mathrm{I})^\times$$

is surjective. But this is the case! For example we know that $\mathbb{G}_m$ is a smooth scheme since it is an open subscheme of $\mathbb{A}^1$ which is obviously smooth. More explicitly: let $\overline{u} \in (\mathrm{A}/\mathrm{I})^\times$ and let u be a lift. Since u reduces to an invertible element, there exists an r such that

$$ru = 1 + j' \qquad j' \in \mathrm{I}.$$

But $1 + j'$ is invertible since $j'^2 = 0$. Hence ru is a unit which means that u must be invertible.

Now, let $\overline{\mathrm{M}} \in \mathrm{GL}_n(\mathrm{A}/\mathrm{I})$, and let M be a lift. We need to prove that M is, in fact, invertible which we can do by checking that $\det(\mathrm{M})$ is invertible; but we know that $\det(\overline{\mathrm{M}})$ is invertible in A/I and $\det(\mathrm{M})$ reduces mod I to $\det(\overline{\mathrm{M}})$ and so the claim reduces to the previous paragraph. □

Exercise 3.3.5. *Prove, via the infinitesimal criterion or other means, that* SL_n *is smooth over any field.*

We will work towards the following theorem:

Theorem 3.3.6. *Let k be a field of characteristic zero, then every affine algebraic k-group is smooth.*

To do so we will discuss the idea of homogeneity. At this point let us *assume that all algebraic groups in sight are finite type over a field.* We restrict ourselves to the following situation: let X be a finite type k-scheme and let $|\mathrm{X}|$ be the underlying topological space of closed points of X; note that the residue field at each closed point of X is a finite extension $\kappa(x)$ of k. We can consider self-maps of k-schemes $f : \mathrm{X} \to \mathrm{X}$ which are invertible; this forms an abstract group (at least for now) which we will denote by $\mathrm{Aut}_k(\mathrm{X})$. In particular, $\mathrm{Aut}_k(\mathrm{X})$ acts on $|\mathrm{X}|$. We say that X is **homogeneous** if this action is transitive on $|\mathrm{X}|$.

Example 3.3.7 (Non-homogeneous varieties). Being a homogeneous variety means that $|\mathrm{X}|/\mathrm{Aut}_k(\mathrm{X}) = *$. If X is positive dimensional, then this cannot happen if the automorphism group is finite. Hurwitz's theorem can be used to compute that this is the case for X a smooth, proper curve of genus $g \geqslant 2$ over an algebraically closed field k. Hence being homogeneous is quite a special property of varieties.

Exercise 3.3.8. *Consider the group scheme μ_m over a field k of characteristic not dividing m. Then if k contains a primitive m-th root of unity prove that*

$$\mu_m \cong \mathbb{Z}/m.$$

In particular show that this group is homogeneous. Assume that k does not contain a primitive cube root of unity (but still charateristic $\neq 3$), then show that μ_3 is not *homogeneous; hint: show that μ_3 is a disjoint union of $*$ and* $\mathrm{Spec}\, k[\mathrm{T}]/(\mathrm{T}^2 + \mathrm{T} + 1)$.

Exercise 3.3.9. *Let k be an algebraically closed field, then* G *is homogeneous.*

The following is immediate from definitions and homogeneity of G; note that smoothness, as explained below, only depends on $\mathrm{G}_{\overline{k}}$.

Proposition 3.3.10. *Let* G *be an algebraic group over k. Then the following are equivalent:*

(1) it is smooth;
(2) the point e is smooth point;
(3) it has at least one smooth point.

Next, we note that the property of being smooth for a group is equivalent to a weaker property: being geometrically reduced. This is not at all surprising; here's the key point and its proof appears in Lemma **??** in the appendix:

LEMMA 3.3.11. *Let k be a perfect field and* X *be a non-empty finite type k-scheme which is reduced. Then there is a dense open subscheme of* X *which is smooth.*

PROPOSITION 3.3.12. *An algebraic group* G *over k is smooth if and only if it is geometrically reduced.*

PROOF. If G is geometrically reduced, then it must have at least one smooth point by Lemma 3.3.11. Therefore Proposition 3.3.10 informs us that G is indeed smooth. ☐

Therefore, to the list of Proposition 3.3.10 we can add: G is geometrically reduced. This is what we will verify later on.

LEMMA 3.3.13. *An algebraic group is smooth if and only if any nilpotent element of* $\mathcal{O}(\mathrm{G})$ *is in* $\mathfrak{m}_e^2$.

PROOF. Because a group G is smooth if and only if it geometrically reduced, the claim follows from the formula

$$\mathrm{T}_e\mathrm{G} = \mathrm{Hom}_k(\mathfrak{m}_e/\mathfrak{m}_e^2, k).$$

☐

LEMMA 3.3.14. *Let* A *be a Hopf algebra, and let* $\mathrm{I} = \ker(\epsilon)$*, i.e., the kernel of augmentation. Then*

1. *As k-vector spaces* $\mathrm{A} = k \oplus \mathrm{I}$*;*
2. *for all $a \in \mathrm{I}$ then a is* ***primitive modulo*** I*, i.e.,*

$$\Delta(a) = a \otimes 1 + 1 \otimes a \mod \mathrm{I} \otimes \mathrm{I}$$

PROOF. These are direct consequences of the properties of Hopf algebras, which we leave for the reader to verify. ☐

THEOREM 3.3.15 (Cartier). *All affine algebraic groups over a field k of characteristic zero is smooth.*

PROOF. First, let us observe that

$$\mathfrak{m}_e = \ker(\epsilon),$$

we denote this maximal ideal as $\mathfrak{m}$. The goal is to use Lemma 3.3.13 to prove that any nilpotent element is in $\mathfrak{m}^2$.

There are two cases; first assume that x maps to zero in $\mathrm{A}_\mathfrak{m}$, then it maps to zero in $\mathrm{A}_\mathfrak{m}/(\mathfrak{m}\mathrm{A}_\mathfrak{m})^2$ and hence zero in $\mathrm{A}/\mathfrak{m}^2$ and hence x is in $\mathfrak{m}^2$.

In the second case, we assume that it is not zero in $\mathrm{A}_\mathfrak{m}$. There then exists an exponent n such that $a^n = 0$ in $\mathrm{A}_\mathfrak{m}$ which is minimal in the sense that $a^{n-1} \neq 0$ in $\mathrm{A}_\mathfrak{m}$. Now, there exists an element s which is not in $\mathfrak{m}$ such that $sa^n = 0$ in A, hence we may assume that $a^n = 0$ in A and $a^{n-1} \neq 0$ in $\mathrm{A}_\mathfrak{m}$. Since $\mathrm{A}/\mathfrak{m} = k$, and has no nilpotents, we also conclude that a must be in $\mathfrak{m}$.

Now, using Lemma 3.3.14 we have that

$$\Delta(a) = a \otimes 1 + 1 \otimes a + y$$

where $y \in \mathfrak{m} \otimes \mathfrak{m}$. Since Δ is a ring map

$$0 = (\Delta(a))^n$$

and so

$$0 = (a \otimes 1 + 1 \otimes a + y)^n.$$

Expanding the right hand side we have terms of the form

$$a^n \otimes 1, \qquad n(a^{n-1} \otimes 1) \cdot (1 \otimes a + y), \qquad (a \otimes 1)^h (1 \otimes a)^i y^j$$

such that $h+i+j = n, i+j \geqslant 2$. Now, the terms of the last form lie in $\mathrm{A} \otimes \mathfrak{m}^2$ while the terms of the last form are zero, hence conclude that

$$n(a^{n-1} \otimes 1) \cdot (1 \otimes a + y) = na^{n-1} \otimes a + n(a^{n-1} \otimes 1)y \in \mathrm{A} \otimes \mathfrak{m}^2$$

so that

$$na^{n-1} \otimes a \in a^{n-1}\mathfrak{m} \otimes \mathrm{A} + \mathrm{A} \otimes \mathfrak{m}^2.$$

Taking reduction modulo $\mathfrak{m}^2$, we contemplate the element

$$na^{n-1} \otimes \overline{a} \in a^{n-1}\mathfrak{m} \otimes \mathrm{A}/\mathfrak{m}^2.$$

The goal is to prove that $\overline{a} = 0$. By the general vector spaces lemma below, it suffices to prove that $na^{n-1} \notin a^{n-1}\mathfrak{m}$. Well there is no way that $a^{n-1} \in a^{n-1}\mathfrak{m}$ because if it were then $a^{n-1} = a^{n-1}m$ where $m \in \mathfrak{m}$ which would mean that $(1-m)a^{n-1}$ but this means that $a^{n-1} = 0$ in $\mathrm{A}_{\mathfrak{m}}$ since $1 - m \in \mathrm{A}_{\mathfrak{m}}^{\times}$. This is a contradiction. Now, n is a unit in A using that A is over a field of characteristic zero. Therefore we conclude that na^{n-1} cannot be in $a^{n-1}\mathfrak{m}$. ☐

LEMMA 3.3.16. *Let* V, V' *be vector spaces over a field and let* W *be a subspace of* V. *Let* y *be a nonzero element of* V'. *Then* $x \in \mathrm{V}$ *lies in* W *if and only if* $x \otimes y$ *lies in* $\mathrm{W} \otimes \mathrm{V}'$.

PROOF. Writing $\mathrm{V} = \mathrm{W} \oplus \mathrm{W}'$ we have that $\mathrm{V} \otimes \mathrm{V}' = (\mathrm{W} \otimes \mathrm{V}') \oplus (\mathrm{W}' \otimes \mathrm{V}')$. Hence $x \in \mathrm{V}$ if and only if $x \otimes y$ avoids the summand $\mathrm{W}' \otimes \mathrm{V}'$. ☐

3.4. Smoothness of Sp_{2n} and orthogonal groups. For the next two exercises, use the lifting criterion:

EXERCISE 3.4.1. *Let k be a field of characteristic not two and let ω be a symplectic form over a field k, then prove that*

$$\mathrm{Sp}(\mathrm{V}, \omega)(\mathrm{A}) \to \mathrm{Sp}(\mathrm{V}, \omega)(\mathrm{A}/\mathrm{I})$$

is surjective for any k-algebra A *and an ideal* I *such that* $\mathrm{I}^2 = 0$. *Describe an isomorphism between the tangent space of* $\mathrm{Sp}(\mathrm{V}, \omega)$ *and* $(\mathrm{Sym}^2 \mathrm{V})^{\vee}$.

EXERCISE 3.4.2. *Let k be a field of characteristic not two; assume the classification of symmetric bilinear forms over an algebraically closed field (they are all equivalent). Prove that the* $\mathrm{O}(\mathrm{V}, q)$*'s are all smooth. Conclude that* $\mathrm{SO}(\mathrm{V}, q)$*'s are also smooth and describe their tangent spaces.*

4. Quotients

We now discuss the idea of taking quotients of groups. We will focus on the case of taking the quotient G/H where H is a normal subgroup of G and sketch the general theory. This is one place where thinking functorially enlightens the situation greatly.

DEFINITION 4.0.1. A morphism of affine group schemes $\mathrm{G} \xrightarrow{q} \mathrm{Q}$ is called a **quotient** if it is faithfully flat.

In particular, the morphism $\mathrm{G} \to \mathrm{Q}$ is surjective. In abstract group theory, a surjective morphism of groups $q : \mathrm{G} \to \mathrm{Q}$ with kernel N satisfies a universal property of sort: given a map of groups $\mathrm{G} \to \mathrm{H}$ with kernel K, we have a map $\mathrm{Q} \to \mathrm{H}$ as soon as K contains the kernel of q. The proof goes something like this: since $\mathrm{G} \to \mathrm{Q}$ is surjective we have that $\mathrm{G}/\mathrm{N} \cong \mathrm{Q}$; so as soon as $\mathrm{N} \leqslant \mathrm{K}$ we do have a map $\mathrm{G}/\mathrm{N} \to \mathrm{H}$. We prove the same result for algebraic groups in the same style.

Theorem 4.0.2. *Let $q : \mathrm{G} \to \mathrm{Q}$ be a quotient morphism with kernel* N*, then any morphism $f : \mathrm{G} \to \mathrm{H}$ whose kernel contains* N *factors uniquely as*

$$\begin{tikzcd} \mathrm{G} \arrow[r, "q"] \arrow[d, "f"] & \mathrm{Q} \arrow[dl, dashed] \\ \mathrm{H} & . \end{tikzcd}$$

If Q was the functor

$$\mathrm{R} \mapsto \mathrm{G}(\mathrm{R})/\mathrm{N}(\mathrm{R})$$

then the result would follow abstract group theory. However this is too naive: the latter functor isn't even a presecheme (in the terminology of these notes). Faithfully flat descent exactly bridges this gap.

Definition 4.0.3. Let $\mathrm{F} : \mathrm{CAlg}_k \to \mathrm{Set}$ be functor and let $\mathrm{G} \subset \mathrm{F}$ be a subfunctor. We say that $\mathrm{G} \subset \mathrm{F}$ is a **flat-equivalence**[2] if for any $x : \operatorname{Spec} \mathrm{R} \to \mathrm{F}$, there exists a faithfully flat map $\operatorname{Spec} \mathrm{R}' \to \operatorname{Spec} \mathrm{R}$ and a filler for the following diagram:

$$\begin{tikzcd} \operatorname{Spec} \mathrm{R}' \arrow[r, dashed] \arrow[d] & \mathrm{G} \arrow[d, hook] \\ \operatorname{Spec} \mathrm{R} \arrow[r, "x"] & \mathrm{F}. \end{tikzcd}$$

Unwinding definitions, this property is saying that for any $x \in \mathrm{F}(\mathrm{R})$, there exists a faithfully flat morphism of rings $\mathrm{R} \to \mathrm{R}'$ such that $x|_{\mathrm{R}'} \in \mathrm{F}(\mathrm{R}')$ is actually in $\mathrm{G}(\mathrm{R}') \subset \mathrm{F}(\mathrm{R}')$.

Remark 4.0.4. We motivate the terminology of "flat-equivalence." We can ask for the stronger lifting property:

$$\begin{tikzcd} & \mathrm{G} \arrow[d, hook] \\ \operatorname{Spec} \mathrm{R} \arrow[r, "x"] \arrow[ur, dashed] & \mathrm{F}. \end{tikzcd}$$

Since G is a subfunctor, this means that $\mathrm{G} \cong \mathrm{F}$ as functors as it would mean that $\mathrm{G} \to \mathrm{F}$ is also an epimorphism. The condition in Definition 4.0.3 is asking that we have an epimorphism *only after* a faithfully flat covering, which is a much weaker but more flexible notion. In this sense, "up to flat covers" G and F are isomorphic.

Example 4.0.5. Consider the multiplicative group $\mathbb{G}_m$; we have the subgroup $\mathbb{G}_m^n \subset \mathbb{G}_m$ consisting of those units which are n-divisible, i.e., x such that there exists a y for which $y^n = x$. For each $x \in \mathrm{R}^\times$ classified by $x : \operatorname{Spec} \mathrm{R} \to \mathbb{G}_m$ we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathrm{R}[\mathrm{T}]/(\mathrm{T}^n - x) \arrow[r, "\mathrm{T}"] \arrow[d] & \mathbb{G}_m^n \arrow[d, hook] \\ \operatorname{Spec} \mathrm{R} \arrow[r, "x"] & \mathbb{G}_m. \end{tikzcd}$$

The key property that we need is following extension lemma:

Lemma 4.0.6. *Let* X *be a scheme and assume that $\mathrm{G} \subset \mathrm{X}$ is a flat-equivalence, then*

1. *any morphism of functors $f : \mathrm{G} \to \mathrm{Y}$ extends uniquely to a morphism of schemes $\mathrm{X} \to \mathrm{Y}$;*
2. *if* G *admits a group structure, then it uniquely extends to a group structure on* X.

---

[2] In Milne's text, this property is called "fat subfunctor"

Proof. We want to take an element $x \in \mathrm{X}(\mathrm{R})$ and map it to $\widetilde{f}(x) \in \mathrm{Y}(\mathrm{R})$. By assumption, we have $\mathrm{R} \to \mathrm{R}'$ a faithfully flat map such that we have the diagram in Definition 4.0.3. Since Y is a scheme, Corollary **??** gives us an equalizer diagram

$$\mathrm{Y}(\mathrm{R}) \to \mathrm{Y}(\mathrm{R}') \rightrightarrows \mathrm{Y}(\mathrm{R}' \otimes_{\mathrm{R}} \mathrm{R}')$$

so that an element of $\mathrm{Y}(\mathrm{R})$ is described as an element of $\mathrm{Y}(\mathrm{R}')$ satisfying the descent condition. Since we have a map $f : \mathrm{G} \to \mathrm{Y}$ and an element $x' \in \mathrm{G}(\mathrm{R}')$ coming from $x \in \mathrm{X}(\mathrm{R})$ we can produce an element $f(x') \in \mathrm{Y}(\mathrm{R}')$ satisfying the descent condition and hence an element in $\mathrm{Y}(\mathrm{R})$. One can check that this element is well defined, i.e., independent of R' (which the reader is encouraged to check). Therefore, we have a well-defined map $\mathrm{X} \to \mathrm{Y}$. Part (2) of the Lemma then follows from part (1). ☐

Remark 4.0.7. In effect, we are proving that a statement of the form:

$$\text{“}a_{\mathrm{fppf}}\mathrm{G} \cong \mathrm{X}.\text{”}$$

Here, a_{fppf} is the functor of sheafification with respect to a **Grothendieck topology**; this is a structure on a category $\mathcal{C}$ (in this case, the category of commutative k-algebras) which axiomatizes what it means to be a covering. From the point of view of the flat topology, Lemma 4.0.6 says that there is no difference between G and X.

Proof of Theorem 4.0.2. Define

$$\widetilde{\mathrm{Q}} : \mathrm{R} \mapsto \mathrm{G}(\mathrm{R})/\mathrm{N}(\mathrm{R});$$

there is a canonical map of functors $\widetilde{\mathrm{Q}} \to \mathrm{Q}$. It suffices to prove that this map is a flat-equivalence. It is a subfunctor: we are looking at the image presheaf of the map $q : \mathrm{G} \to \mathrm{Q}$. To prove that it is indeed a flat equivalence, we consider the following commutative diagram where the outer square is cartesian:

$$\begin{tikzcd}
\mathrm{G} \arrow[d] & \mathrm{G} \times_{\mathrm{Q}} \operatorname{Spec} \mathrm{R} \arrow[l] \arrow[ld, dashed] \arrow[dd] \\
\widetilde{\mathrm{Q}} \arrow[d] & \\
\mathrm{Q} & \operatorname{Spec} \mathrm{R}. \arrow[l, "x"]
\end{tikzcd}$$

Note that the fiber product is an affine scheme and the map to $\operatorname{Spec} \mathrm{R}$ is faithfully flat. The dashed arrows then witnesses the fact that $\widetilde{\mathrm{Q}} \to \mathrm{Q}$ is a flat-equivalence. ☐

4.1. Existence of quotients. Next, we address the existence of quotients. If $\mathrm{G} \to \mathrm{Q}$ is a quotient, then the kernel is a normal subgroup. Indeed, the kernel is taken pointwise and the result follows from abstract group theory. Going the other way is more subtle and requires us to construct the quotient $\mathrm{G} \to \mathrm{Q}$.

Theorem 4.1.1. *Let* $\mathrm{N} \leqslant \mathrm{G}$ *be a subgroup of affine algebraic groups. Then, if* $\mathrm{N} \leqslant \mathrm{G}$ *is a normal subgroup then there exists a quotient map* $\mathrm{G} \to \mathrm{Q}$ *whose kernel is exactly* N.

The strategy to prove Theorem 4.0.2 is to first realize N as the kernel of a representation $\mathrm{G} \to \mathrm{GL}(\mathrm{V})$. This part is representation-theoretic in nature, though the map G to GL(V) will almost always not be surjective. Next, we can then appeal to a structural result about group homomorphisms $\mathrm{G} \to \mathrm{H}$ which states that any such morphism factors as a quotient followed by a closed immersion. Let us address the representation-theoretic part of the story. If $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ is a finite dimensional representation and $\mathrm{W} \subset \mathrm{V}$ is a subspace, then the **stabilizer** $\mathrm{G}_{\mathrm{W}} \subset \mathrm{G}$ is the functor

$$\mathrm{G}_{\mathrm{W}} : \mathrm{R} \mapsto \{g \in \mathrm{G}(\mathrm{R}) : g(\mathrm{W} \otimes_k \mathrm{R}) = \mathrm{W} \otimes_k \mathrm{R}\}$$

Lemma 4.1.2. *The stabilizer group* $\mathrm{G}_{\mathrm{W}} \leqslant \mathrm{G}$ *is a subgroup.*

Proof. The idea of the proof will be similar to the more interesting Chevalley's theorem so let's give a proof. Let

$$\rho : \mathcal{O}(G) \to V \otimes \mathcal{O}(G)$$

be the comodule corrresponding to ρ. Let $(e_i)_{I_W}$ be a basis for W which we can extend to a basis for V which we write as $(e_i)_{I_W \sqcup J}$. We can write

$$\rho(e_i) = \sum_{j \in I_W \sqcup J} e_j \otimes a_{ji}.$$

Now, let R be an arbitrary k-algebra, then an element $g \in G(R)$ is the same thing as map $g : \mathcal{O}(G) \to R$. We can describe the action of

$$g\cdot : V_R \to V_R$$

in the following way: it acts on $e_i = e_i \otimes 1 \in V_R$ as

$$g(e_i) = \sum_{j \in I_W \sqcup J} e_j \otimes g(a_{ji}).$$

Therefore we conclude that $g(W \otimes R) \subset W \otimes R$ if and only if g annihilates a_{ij} for $i \in I_W$ and $j \in J$. Therefore, we see that G_W is represented by the ring $\mathcal{O}(G)/(a_{ji}, i \in I_W, j \in J)$. $\square$

Theorem 4.1.3 (Chevalley). *Let* G *be an affine algebraic group, then for any subgroup* $H \leqslant G$ *there exists a representation* $\rho : G \to GL(V)$ *and a* 1*-dimensional subspace* $L \subset V$ *such that* $H \cong G_L$.

Proof. Let I be the kernel of the surjective map $\mathcal{O}(G) \to \mathcal{O}(H)$; this is a finitely generated ideal. Then there exists a finite dimensional subspace V of $\mathcal{O}(G)$ containing the generating set of I for which

$$\Delta(V) \subset V \otimes \mathcal{O}(G).$$

Let us set $W := I \cap V$ with basis $(e_j)_{j \in J_W}$ which we extend to a basis $(e_j)_{j \in K \sqcup J_W}$. Let

$$\Delta e_j = \sum_{i \in K \sqcup J_W} e_i \otimes a_{ij}.$$

We note that $\mathcal{O}(G_W) \cong \mathcal{O}(G)/(a_{ij} : i \in K, j \in J_W)$.

Set $I' := (a_{ij} : i \in K, j \in J_W)$. We now make two claims: (1) that $I' \subset I$ and that (2) $e_j = \sum_{i \in I} \epsilon(a_{ij}) e_i$ for $j \in J_W$. Having these two claims we see that $I' = I$ since the e_j's generate I. Hence we must have that $H = G_W$. At this point we have furnished a subspace of V for which H is its stabilizer. To get a one dimensional subspace we simply replace W by the determinant $\wedge^{\dim(W)} W = \det(W)$.

To prove claim 1) we note that

$$\Delta(I) \subset \mathcal{O}(G) \otimes I + I \otimes \mathcal{O}(G)$$

because the map $\mathcal{O}(G) \to \mathcal{O}(H)$ is a map of Hopf algebras and I was declared to be the kernel. This means that the $a_{ij} \in I$.

To prove claim 2), we note that $\epsilon(I) = 0$ again because $\mathcal{O}(G) \to \mathcal{O}(H)$ is a map of Hopf algebras. Hence $e_j = (\epsilon \otimes \mathrm{id})(\Delta)(e_j) = \sum_{i \in I} \epsilon(a_i j) e_i$ as claimed. $\square$

Chevalley's theorem lets us express H as the stabilizer of a representation. We want to do a bit better: we want to say that H acts via the trivial character. This is to say that we have a commutative diagram. If $H \leqslant G$ is a subgroup, we say that a character $\chi : H \to GL_1$ **occurs** in a representation of G if there exists a representation $\rho : G \to GL(V)$ and a 1-dimensional subspace $L \subset V$, such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\rho"] & GL(V) \\
H \arrow[u] \arrow[r, dashed, "\chi"] & GL_1 = GL(L). \arrow[u]
\end{tikzcd}$$

We also recall that the trivial character $\chi : \mathrm{H} \to \mathrm{GL}_1$ is the character that sends everything to 1, i.e., one that factors through $\mathrm{H} \to \{1\} \hookrightarrow \mathrm{GL}_1$.

Exercise 4.1.4. *Prove that a character $\chi : \mathrm{G} \to \mathbb{G}_m$ is the same datum as an invertible element $a \in \mathcal{O}(\mathrm{G})$ such that $\Delta(a) = a \otimes a$. Let a be such an element so that it admits an inverse a^{-1}; prove that the corresponding character is the dual representation, which we write as $-\chi$.*

Lemma 4.1.5. *With the set-up of Theorem 4.1.3 and assume:*

1. *(1) over an algebraically closed field k,*
2. *(2) H is a normal subgroup.*

then, it is possible to choose L in Theorem 4.1.3 such that H acts via the trivial character.

Proof sketch. Let L be as in the conclusion of Theorem 4.1.3, we first claim that over an algebraically closed field k there exists $m \gg 0$ such that $\mathrm{L}^{-\otimes m}$ occurs in a representation of G. The idea is quite simple and m can, in good cases, be chosen to just be 1. If $\mathrm{L} \subset \mathrm{V}$ is a 1-dimensional subspace as in Theorem 4.1.3 then $\mathrm{L}^{\vee}$ is the obvious space on which H acts by $-\chi$ (as in the previous exercise). However, $\mathrm{V}^{\vee} \to \mathrm{L}^{\vee}$ is not necessarily split as an H-representation and hence we cannot say that $\mathrm{L}^{\vee}$ occurs directly in $\mathrm{V}^{\vee}$. What would suffice is if we can arrange L to be a *direct summand* of V as an H-representation. To do so, we use Lemma **??**; we arrange L to lie in a semisimple H-representation which also happens to have an extension to a G-structure.

We do the obvious thing: let W be the sum of all 1-dimensional subspaces of V which are H-stable. This is clearly a semisimple H-representation containing L as a subobject. But now L must be a summand as an H-representation. We will be done if we can prove that W is G-stable. To prove G-stability we consider $g \in \mathrm{G}(k)$ and the subspace $g\mathrm{D}$. In this case, $g\mathrm{D}$ is stable under $g\mathrm{H}g^{-1} = \mathrm{H}$ (by normality) and hence W is stable under $\mathrm{G}(k)$. To promote this to G-stability we use the concept of density as in Section **??**. Indeed if $\mathrm{G}(k)$ is schematically dense then a $\mathrm{G}(k)$-stable subspace W must be G-stable because G_{W} is a closed subscheme by Lemma 4.1.2. We are thus done whenever G is smooth since $\mathrm{G}(k)$ is geometrically reduced by Theorem **??**. Indeed, this doesn't quite work for non-smooth groups, but the result is still true under tensoring L further by some power (see [**?**, Lemma 5.16] for details)

Having this claim, we can just take the tensor powers of appropriate representation and note that $\mathrm{L}^{\otimes m} \otimes \mathrm{L}^{\otimes -m}$ occurs in this large representation and thus H acts via the trivial character. □

We set one more notation: if $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$, then we have the subspace of V fixed by a group as

$$\mathrm{V}^{\mathrm{G}} = \{v \in \mathrm{G} : gv_{\mathrm{R}} = v_{\mathrm{R}}, \forall \mathrm{R} \in \mathrm{CAlg}_k\}.$$

This is also computed in the language of comodules as

$$\mathrm{V}^{\mathrm{G}} = \{v \in \mathrm{V} : \rho(v) = v \otimes 1\}.$$

Corollary 4.1.6. *Over an algebraically closed field, any normal subgroup H of G can be presented as the kernel of a representation, i.e., we have an exact sequence of groups:*

$$1 \to \mathrm{H} \to \mathrm{G} \xrightarrow{\rho} \mathrm{GL}(\mathrm{V}).$$

Proof. By Lemma 4.1.5 we can find a G-representation such that H acts via the trivial representation. In this case, since H is normal, G stabilizes the H-fixed vectors:

$$n(g \cdot v) = ng \cdot v = gn' \cdot v = g \cdot v \qquad n, n' \in \mathrm{H}, g \in \mathrm{G}, v \in \mathrm{V},$$

hence we have a representation of G on V^{H}. Let K be the kernel of this representation, and we claim that $\mathrm{K} = \mathrm{H}$. It is obvious that $\mathrm{H} \subset \mathrm{K}$ because we are looking at G acting on V^{H}. But now, $\mathrm{L} \subset \mathrm{V}^{\mathrm{H}}$ since H acts on L via the trivial character and k acts trivially on V^{H} by definition. Hence $\mathrm{K} \subset \mathrm{G}_{\mathrm{L}}$ but, then $\mathrm{G}_{\mathrm{L}} = \mathrm{H}$ as in Theorem 4.1.3.

□

Theorem 4.1.7 (Factorization). *Any morphism of affine algebraic groups* $\mathrm{G} \to \mathrm{H}$ *factors as*

$$\mathrm{G} \xrightarrow{q} \mathrm{Q} \xrightarrow{i} \mathrm{H}$$

where q *is a quotient and* i *is a closed immersion.*

We will not go into the proof of Theorem 4.1.7 but the main point is [?][Theorem 3.31] which states that any inclusion of finitely generated Hopf algebras is, in fact, a faithfully flat map. The factorization $\mathrm{G} \to \mathrm{Q} \to \mathrm{H}$ is then induced by a factorization

$$\mathcal{O}(\mathrm{H}) \to \mathrm{C} \to \mathcal{O}(\mathrm{G})$$

where the first map is surjective and the second map is injective.

Proof of Theorem 4.1.1. Consider the base change $\mathrm{N}_{\overline{k}} \to \mathrm{G}_{\overline{k}}$ to the closure. We can express $\mathrm{N}_{\overline{k}}$ as the kernel of some map of groups $\mathrm{G}_{\overline{k}} \to \mathrm{H}_{\overline{k}}$ by Corollary 4.1.6. In fact, we can choose k'/k a finite extension where this already happens. Hence we have maps

$$\mathrm{G} \to \mathrm{G}_{k'} \to \mathrm{H}_{k'};$$

which on an k-algebra R takes the form

$$\mathrm{G} \to \mathrm{G}(\mathrm{R}_{k'}) \to \mathrm{H}(\mathrm{R}_{k'})$$

Now, the first map is injective since $\mathrm{R} \to \mathrm{R}_{k'}$ is faithfully flat, hence the kernel of the composite is computed as the intersection of G(R) with $\mathrm{N}(\mathrm{R}_{k'})$ which is N(R) itself again because $\mathrm{R} \to \mathrm{R}_{k'}$ is faithfully flat. Therefore, N is the kernel of the composite $\mathrm{G} \to \mathrm{H}_{k'}$. We can thus use Theorem 4.1.7 to factor it as a map $\mathrm{G} \xrightarrow{q} \mathrm{Q} \to \mathrm{H}_{k'}$ where q is faithfully flat with kernel N. □

4.2. Quotients in general. We first pin down the idea of a quotient G/H where H is not necessarily normal.

Definition 4.2.1 (Quotient). Let H be a subgroup of G, then a **quotient** of G by H is a finite type scheme X equipped with an action

$$\mu : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$$

and a base, rational point $o \in \mathrm{X}(k)$ such that

$$\mu_o : \mathrm{G} \to \mathrm{X} \qquad g \mapsto g \cdot o$$

realizes $\mathrm{G}/\mathrm{H} \to \mathrm{X}$ as a flat-equivalence

This is a bit wordy, but the point is that G/H is flat-equivalent to X. To define maps out of X is not so hard: assume that we have a morphism of schemes $\mathrm{G} \to \mathrm{Y}$ which is constant on cosets of H(R) in G(R) for all R, then there is a unique map $\mathrm{X} \to \mathrm{Y}$ under μ_o [**?**, Proposition 5.21]. This uses the fact that G/H is flat equivalent to X.

To get a better sense of quotients it is useful to explain the general situation of a group acting on a scheme X. Suppose that we have such a situation $\mu : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$, then for any $x \in \mathrm{X}(k)$ we have the **orbit map**

$$\mu_x : \mathrm{G} \to \mathrm{X} \qquad g \mapsto gx;$$

which is locally closed [**?**, Proposition 1.65]. The **orbit** of x is then the reduced image

$$\mu_x(\mathrm{G}) \hookrightarrow \mathrm{X}.$$

Again with $x \in \mathrm{X}(k)$ we have the **isotropy** of G at x which is the subscheme G_x defined via pullback:

$$\begin{tikzcd}
\mathrm{G}_x \arrow[r] \arrow[d] & \mathrm{G} \arrow[d, "\mu_x"] \\
\operatorname{Spec} k \arrow[r, "x"] & \mathrm{X}.
\end{tikzcd}$$

The isotropy of G at x admits the following description:

$$\mathrm{G}_x(\mathrm{R}) = \{g \in \mathrm{G}(\mathrm{R}) : gx|_{\mathrm{R}} = x|_{\mathrm{R}}\};$$

via the pullback description it is obviously a subgroup of G and, in particular, a scheme. To prove that G/H exists, one employs a similar idea as in the case of normal subgroups.

PROPOSITION 4.2.2. *Let* $\mu : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ *be an action and assume:*

(1) G *is smooth;*
(2) X *is separated;*

and fix $x \in \mathrm{X}(k)$*. then the quotient* G/G_x *exists and the map* $\mathrm{G}/\mathrm{G}_x \to \mathrm{X}$ *is an immersion.*

We refer to [**?**, Theorem 7.18] for details:

THEOREM 4.2.3. *If* G *is a smooth and affine, then* G/H *exists as a separated scheme of finite type over* k*.*

PROOF SKETCH. □

CHAPTER 3

The Tannakian philosophy

Let us recall the following idea from Fourier theory: given a square integrable function $f : \mathrm{S}^1 \to \mathbb{C}$ we can extract from it a sequence of numbers, its **Fourier coefficients**:

$$c_n := \int_{\mathrm{S}^1} f(t)e^{-2\pi int}dt.$$

Basic Hilbert space theory applied to the space of square-integrable function tells us that we can recover f from its Fourier coefficients. Regarding c_n as a $\mathbb{Z}$-sequence of integers, we can regard the above result as an identification between the space $\mathrm{L}^2(\mathrm{S}^1)$ (with its standard Haar measure) and $\ell^2(\mathbb{Z})$. Under this identification, the function on S^1 given by $\chi_n(t) := e^{2\pi int}$, which form an orthonormal basis for $\mathrm{L}^2(\mathrm{S}^1)$, is swapped with the Dirac delta function δ_n which places 1 at the n-th place and zero otherwise. This is part of abstract harmonic analysis: we start with a topological group G, like S^1 and look at its **Pontryagin dual** computed as the space of continuous characters $\widehat{\mathrm{S}}^1 := \mathrm{Hom}_{\mathrm{cts}}(\mathrm{S}^1, \mathrm{S}^1)$ which recovers the topological group $\mathbb{Z}$. One of the miracles of abstract harmonic analysis is that the Pontryagin dual of a compact abelian group is discrete so the study of them is equivalent to the study of discrete groups. This allows for conversion of problems which may be difficult in one and easy in the other. The following question is then natural:

Question 0.0.1. What is the analog of Pontryagin duality for nonabelian groups?

Abstractly, Pontryagin duality is the statement that a compact abelian group may be recovered from its group of characters; this isn't going to be enough for nonabelian groups in general. Tannaka duality provides a setting where this is possible: the group can be recovered from its category of representations. This is a wildly powerful idea in modern mathematics: we can recover a widget X by understanding the various ways that X can act on "linear" gadgets.

1. Representation theory

We work over a field k; we also recall the standing assumption that G is of finite type over k. Let us begin with some basic definitions. A **linear representation** of an affine algebraic group G is a group homomorphism

$$\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}),$$

where V is a k-vector space V; strictly speaking this is a morphism between group objects in functors from CAlg_k to Set. If V is finite dimensional, then clearly GL(V) is a scheme and we are looking at an honest morphism of algebraic groups. One way to think about a linear representation is via functor-of-points: for each k-algebra R we are given an action of G(R) on $\mathrm{V} \otimes \mathrm{R}$, i.e. a group homomorphism

$$\rho_{\mathrm{R}} : \mathrm{G}(\mathrm{R}) \to \mathrm{Aut}_{\mathrm{R}}(\mathrm{V}_{\mathrm{R}}).$$

These maps are subject to compatibilities which the reader should take time to unpack. The fact that G is affine, however, lets us package all of this information in a more economical way. As already explained above, $\mathcal{O}(\mathrm{G})$ is a Hopf-algebra with comultiplication we denote by Δ; a **right $\mathcal{O}(\mathrm{G})$-comodule** is a k-linear map

$$\mathrm{V} \to \mathrm{V} \otimes_k \mathcal{O}(\mathrm{G});$$

such that the following diagrams commute:

$$\begin{tikzcd}
\mathrm{V} \arrow[r, "\rho"] \arrow[d, "\rho"'] & \mathrm{V} \otimes_k \mathcal{O}(\mathrm{G}) \arrow[d, "\mathrm{id}\otimes\Delta"] \\
\mathrm{V} \otimes_k \mathcal{O}(\mathrm{G}) \arrow[r, "\rho\otimes\mathrm{id}"] & \mathrm{V} \otimes_k \mathcal{O}(\mathrm{G}) \otimes_k \mathcal{O}(\mathrm{G}).
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{V} \arrow[r, "\rho"] \arrow[dr, "\mathrm{id}"'] & \mathrm{V} \otimes_k \mathcal{O}(\mathrm{G}) \arrow[d, "\mathrm{id}\otimes\epsilon"] \\
& \mathrm{V}.
\end{tikzcd}$$

Remark 1.0.1. Comodules and representations are two equivalent pieces of structure. Indeed, given a representation, we have a tautological element $a \in \mathrm{G}(\mathcal{O}(\mathrm{G}))$ corresponding to the identity; this gets map to an element

$$\rho_{\mathcal{O}(\mathrm{G})}(a) \in \mathrm{End}_{\mathcal{O}(\mathrm{G})}(\mathrm{V} \otimes \mathcal{O}(\mathrm{G})).$$

The map $\rho_{\mathcal{O}(\mathrm{G})}(a)$ is equivalent to a k-linear map

$$\rho_{\mathcal{O}(\mathrm{G})}(a) : \mathrm{V} \to \mathrm{V} \otimes \mathcal{O}(\mathrm{G}).$$

Exercise 1.0.2. *Check that this defines a comodule structure on* V.

On the other hand, if we are given a comodule structure $\rho : \mathrm{V} \to \mathrm{V} \otimes \mathcal{O}(\mathrm{G})$, we want to construct a morphism $\mathrm{G} \to \mathrm{GL}(\mathrm{V})$. For any morphism

$$f \in \mathrm{G}(\mathrm{R}) \qquad f : \mathcal{O}(\mathrm{G}) \to \mathrm{R},$$

we get an automorphism of V_{R} as follows:

$$\mathrm{V}_{\mathrm{R}} \xrightarrow{\rho\otimes\mathrm{id}} \mathrm{V} \otimes \mathcal{O}(\mathrm{G}) \otimes \mathrm{R} \xrightarrow{\mathrm{id}\otimes f\otimes\mathrm{id}} \mathrm{V} \otimes \mathrm{R} \otimes \mathrm{R} \xrightarrow{\mathrm{id}\otimes m} \mathrm{V}.$$

From now on, we will abuse this equivalence and write comodule structures using the same Greek letters as representations: $\rho : \mathcal{O}(\mathrm{G}) \to \mathrm{V} \otimes_k \mathcal{O}(\mathrm{G})$.

Remark 1.0.3. There is a more "equational" way to understand comodule structures: pick a basis for a finite dimensional vector space V, $e_1, \cdots, e_n$ and suppose that we are given a k-linear map

$$\rho : \mathrm{V} \to \mathrm{V} \otimes \mathcal{O}(\mathrm{G}).$$

Then we can generically write it as

$$\rho(e_j) = \sum e_i \otimes a_{ij},$$

where $a_{ij} \in \mathcal{O}(\mathrm{G})$. Unwinding the definition of a comodule leads us to conclude that ρ is a comodule structure if and only if:

(1) $\Delta(a_{ij}) = \sum_\ell a_{i\ell} \otimes a_{\ell j}$
(2) $\epsilon(a_{ij}) = \delta_{ij}$, where δ_{ij} is the usual indicator function.

Under the correspondence above we can also be more explicit about the corresponding representation: $\mathrm{G} \to \mathrm{GL}(\mathrm{V})$. On k-points $g \in \mathrm{G}(k)$ is sent to the automorphism described by

$$e_j \mapsto \sum e_i a_{ij}(g).$$

The reader is also encouraged to check that the map on function $\mathcal{O}(\mathrm{GL}_{\mathrm{V}}) \to \mathcal{O}(\mathrm{G})$ is given by sending T_{ij} to a_{ij}.

Example 1.0.4. The **standard representation** of GL(V) on the vector space V can be described as $\mathrm{GL}(\mathrm{V}_{\mathrm{R}})$ on V_{R} via functor-of-points. What is the coaction? Well we need to provide a coaction map

$$\rho : \mathrm{V} \to \mathrm{V} \otimes \mathcal{O}(\mathrm{GL}(\mathrm{V}));$$

after choosing a basis for V (say $e_1, \cdots, e_n$) this can be written down explicitly:

$$\rho : \mathrm{V} \to \mathrm{V} \otimes k[\mathrm{T}_{ij}, \det^{-1}] \qquad e_j \mapsto \sum_{1 \leqslant i \leqslant n} e_i \otimes \mathrm{T}_{ij}.$$

1.1. Linearity of algebraic groups. One of the most pleasant aspects of the theory of affine algebraic groups is that they are all linear: subgroups of GL(V) for some finite dimensional vector space V. This means that we can choose coordinates on G and reduce a lot of problems to problems about matrices.

Definition 1.1.1. A representation ρ is said to be **faithful** if the maps

$$\rho_{\mathrm{R}} : \mathrm{G}(\mathrm{R}) \to \mathrm{GL}(\mathrm{V}_{\mathrm{R}})$$

are all injective.

Definition 1.1.2. A group G is **linear** if there exists a finite dimensional vector space V and a faithful representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$.

Remark 1.1.3. In fact, one can prove that any faithful representation ρ defines a closed immersion $\mathrm{G} \hookrightarrow \mathrm{GL}(\mathrm{V})$. This means that we can realize G as a subgroup of GL(V). We might prove this later on.

We will soon prove that affine algebraic groups are linear via the **regular representation**. At this point it is useful to introduce the notion of a group action on a scheme.

Definition 1.1.4. Let X be a k-scheme, then a **left G-action** on X is a morphism $\mathrm{G} \times \mathrm{X} \to \mathrm{X}$ such that for any R

$$\mathrm{G}(\mathrm{R}) \times \mathrm{X}(\mathrm{R}) \to \mathrm{X}(\mathrm{R})$$

defines a G(R) action on X(R).

If X is affine, then an action is the same thing as a ring map

$$\mathcal{O}(\mathrm{X}) \to \mathcal{O}(\mathrm{X}) \otimes \mathcal{O}(\mathrm{G})$$

such that the map on underlying k-vector spaces define an $\mathcal{O}(\mathrm{G})$-comodule structure on $\mathcal{O}(\mathrm{X})$. Of course $\mathcal{O}(\mathrm{X})$ tends to be infinite dimensional in general, as a k-vector space. However the action is, in some sense, "locally finite."

Lemma 1.1.5. *Let* V *be a (possibly infinite dimensional) representation of an affine algebraic group* G *over* k. *Then every finite-dimensional* k*-subspace of* V *is contained in some finite-dimensional subrepresentation of* V.

Proof. Let $\{e_\alpha\}$ be a basis for $\mathcal{O}(\mathrm{G})$ and define $a_{\alpha\beta} \in \mathcal{O}(\mathrm{G})$ by

$$\Delta(e_\alpha) = \sum_\alpha a_{\alpha\beta} \otimes e_\beta.$$

(Note that for a fixed β there are only finitely many α with $a_{\alpha\beta} \neq 0$). Now let $\{v_i\}$ be a basis for our finite-dimensional vector subspace V_0 and let define $w_{i\alpha}$ by

$$\rho(v_i) = \sum_\alpha w_{i\alpha} \otimes e_\alpha.$$

Note again that only finitely many $w_{i\alpha}$ are non-zero, and let W be the span of these $w_{i\alpha}$. We claim that this is a subrepresentation of V which contains V_0. To see that W contains V_0, note that

$$v_i = (id \otimes \epsilon) \circ \rho(v_i) = \sum_\alpha \epsilon(e_\alpha) w_{i\alpha}.$$

To see that W is a subrepresentation of V, note that

$$(\rho \otimes id) \circ \rho(v_i) = \sum_\alpha \rho(w_{i\alpha}) \otimes e_\alpha$$

is equal to

$$(id \otimes \Delta) \circ \rho(v_i) = \sum_{\alpha} w_{i\alpha} \otimes \Delta(e_\alpha) = \sum_{\alpha,\beta} w_{i\alpha} \otimes a_{\alpha\beta} \otimes e_\beta = \sum_{\alpha,\beta} w_{i\beta} \otimes a_{\beta\alpha} \otimes e_\alpha.$$

(For the last equality we just reindexed the sum.) Since $\{e_\alpha\}$ is a basis for $\mathcal{O}(\mathrm{G})$ this implies that $\rho(w_{i\alpha}) = \sum_\beta w_{i\beta} \otimes a_{\beta\alpha}$. This shows that $\rho(\mathrm{W}) \subseteq \mathrm{W} \otimes \mathcal{O}(\mathrm{G})$, i.e. that W is a subrepresentation of V. ☐

Remark 1.1.6. Here's another way to interpret Lemma 1.1.5 via k-points. We see that $\mathrm{G}(k)$ acts on $\mathcal{O}(\mathrm{X})$: given a $g \in \mathrm{G}(k)$ then it does:

$$f \mapsto (g \cdot f)(x) = f(g(x)).$$

But now, in the notation of the proof, we see that the function $g \cdot f$ is just $\sum h_i(g)f$. So Lemma 1.1.5 says that the $\mathrm{G}(k)$-action on $\mathcal{O}(\mathrm{X})$ is such that any point is contained in a finite dimensional k-vector space of $\mathcal{O}(\mathrm{X})$ which is $\mathrm{G}(k)$**-invariant**.

Theorem 1.1.7. *Let* G *be an affine algebraic group over k, then there exists a finite dimensional vector space* V *such that* G *is a subgroup of* GL(V)*. That is, affine algebraic groups are linear.*

Proof of Theorem 1.1.7. The comultiplication $\Delta : \mathcal{O}(\mathrm{G}) \to \mathcal{O}(\mathrm{G}) \otimes_k \mathcal{O}(\mathrm{G})$ defines an $\mathcal{O}(\mathrm{G})$-comodule structure on $\mathcal{O}(\mathrm{G})$. Of course $\mathcal{O}(\mathrm{G})$ is far from being finite dimensional as a vector space over k but, by Lemma 1.1.5 we can find V, a finite dimensional (as a k-vector space) A-comodule which contains the generators of $\mathcal{O}(\mathrm{G})$ as a k-algebra; we write these as $e_1, \cdots e_n$. We then have a map of schemes $\mathrm{G} \to \mathrm{GL}(\mathrm{V})$. The goal is to prove that this map is a closed immersion, which is to say that the map

$$\mathcal{O}(\mathrm{GL}(\mathrm{V})) \to \mathcal{O}(\mathrm{G})$$

is surjective. Writing the comultiplication on e_j as

$$\Delta(e_j) = \sum e_i \otimes a_{ij},$$

we see that the image of $\mathcal{O}(\mathrm{GL}(\mathrm{V})) \to \mathcal{O}(\mathrm{G})$ contains the a_{ij}'s (because this map is $\mathrm{T}_{ij} \mapsto a_{ij}$). We claim that, in fact, the image contains e_j's. This follows from the identity axiom:

$$e_j = (\epsilon \otimes \mathrm{id})(\Delta(e_j)) = \sum \epsilon(e_j) a_{ij}.$$

Hence we conclude that the image contains V; but V generates A as a k-algebra and the above map is a k-algebra map. We conclude that it must be surjective. ☐

In other words, algebraic groups are linear.

1.2. Graded vector spaces versus $\mathbb{G}_m$-representations. Before we embark on representation theory proper, we will discuss an example of the process of *geometrization* which has become a really powerful idea. It is one of those things which does not really have a precise definition but one knows it when one sees it. We want to discuss the notion of a graded vector space. If k is a field, then a graded vector space V is just the datum of a decomposition of V into k-vector spaces:

$$\mathrm{V} \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{V}(i).$$

This structure occurs all over mathematics — from Hodge theory, to algebraic topology and even in combinatorics. I want to explain the following result:

Proposition 1.2.1. *There is an equivalence of categories between $\mathbb{G}_m$-representations over k and k-vector spaces.*

Let us denote the category of $\mathbb{G}_m$-representations are $\mathrm{Rep}(\mathbb{G}_m)$; morphisms in this category are maps of k-vector spaces $\mathrm{V} \to \mathrm{W}$ which commute with the $\mathbb{G}_m$-action. On the other hand, the category of graded vector spaces is denoted grVect_k; note that the category of graded vector spaces is a functor category of functors from the discrete category $\mathbb{Z}^\delta$ to Vect_k. The morphisms are morphisms of vector spaces which preserve grading. One of the points of this result is that the reader should appreciate what it means to give a k-linear map

$$\mathrm{V} \to \mathrm{V} \otimes \mathcal{O}(\mathbb{G}_m) = \mathrm{V}[t, t^{-1}].$$

Proof. First we construct a functor:

$$\mathrm{Rep}(\mathbb{G}_m) \to \mathrm{grVect}_k.$$

We break this down into several observations. First, observe that we have an isomorphism of k-vector spaces:

$$\mathrm{V} \otimes_k k[t, t^{-1}] \cong \bigoplus_{j \in \mathbf{Z}} \mathrm{V}(j),$$

where we set (this is a convention we choose) that

$$\mathrm{V}(j) = \mathrm{V}\{t^{-j}\}.$$

Explicitly we write elements in $\mathrm{V}(j)$ as vt^{-j} where $v \in \mathrm{V}$ and so the above map can be described as

$$\rho : \mathrm{V} \to \bigoplus_{j \in \mathbf{Z}} \mathrm{V}(j) \qquad f \mapsto (\rho_j(f) \in \mathrm{V}(j)) = \sum \rho_j(f) t^{-j}.$$

We note that the direct sum indicates that the components of $(\rho_j(f))$ is finitely supported, i.e., zero except for finitely many j's.

The second observation is that any $f \in \mathrm{V}$ can actually be written uniquely as

$$f = \sum \rho_j(f).$$

This is a consequence of the identity axiom. Indeed, we note that the augmentation $\mathrm{V} \otimes k[t, t^{-1}] \to \mathrm{V}$ is given by "evaluation at 1" so, in the direct sum presentation, the map

$$\bigoplus \mathrm{V}(j) \to \mathrm{V}$$

is given by $(f_j) \mapsto \left(\sum f_j t^{-j}\right)_{t=1} = \sum f_j$. Since the identity axiom dictates the composite

$$\mathrm{V} \to \bigoplus_{j \in \mathbf{Z}} \mathrm{V}(j) \xrightarrow{t \mapsto 1} \mathrm{V}$$

must be the identity, we get that for any $f \in \mathrm{V}$

$$f = \sum_{j \in \mathbf{Z}} \rho_j(f),$$

so that any f can be uniquely written as a finite sum of the $\rho_j(f)$'s.

The third observation is that ρ is, in a sense, idempotent:

$$\rho(\rho_j(f)) = \rho_j(f).$$

Let us prove this first. Coassociativity of the comodule structure gives the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{V} \arrow[r, "\rho"] \arrow[d, "\rho"] & \bigoplus_{j \in \mathbf{Z}} \mathrm{V}(j) \arrow[d, "\rho"] \\
\bigoplus_{j \in \mathbf{Z}} \mathrm{V}(j) \arrow[r, "\Delta"] & \bigoplus_{j \in \mathbf{Z}} \bigoplus_{k \in \mathbf{Z}} \mathrm{V}(jk).
\end{tikzcd}$$

In coordinates: let $f \in \mathrm{V}$; generically we can write $\rho(f) = \sum_i \rho_i(f) t^{-i}$ then going to the right and down gives:

$$\rho(\rho(f)) = \rho(\sum_i \rho_i(f) t^{-i}) = \sum_i \rho(\rho_i(f)) t^{-i},$$

while going down and then left gives

$$\Delta(\rho(f)) = \Delta(\sum_i \rho_i(f)t^{-i}) = \sum_i \rho_i(f)\Delta(t^{-i}) = \sum_i \rho_i(f)t^{-i}u^{-i};$$

so that, comparing coefficients, we conclude.

At this point we can define a grading on V by saying that f is homogeneous of degree i if and only if $\rho(f) = ft^{-i}$. The well-definedness of this definition is observation 3 and the fact that any element in V can be written as a sum of homogeneous pieces is observation 2.

We leave the reader to define, out of a grading, a $\mathbb{G}_m$-action where the equivalence will become obvious.

□

So what are we geometrizing? In effect, we are giving a geometric interpretation of the category of graded vector spaces: it is the algebro-geometric category of $\mathbb{G}_m$-representations. In fact, one can do slightly better using the language of **stacks** which we will not explore in this class (but is extremely useful in representation theory). The category of $\mathbb{G}_m$-representations is the same thing as the category of quasicoherent sheaves over the classifying stack $\mathrm{B}\mathbb{G}_m$.

EXERCISE 1.2.2. *Let* A *be an abelian group and let* A^{ds} *be the category whose objects are elements of* A *and there is no non-trivial morphisms. The category of* A***-graded vector spaces*** *is the category* $\mathrm{Fun}(\mathrm{A}^{\mathrm{ds}}, \mathrm{Vect}_k)$.

(1) *The group algebra of* A *is the algebra* $k[\mathrm{A}]$ *whose elements are of the form*

$$\sum_{a\in \mathrm{A}} c_a a \qquad \textit{the sum is finite},$$

and addition is given componentwise

$$\sum_{a\in \mathrm{A}} c_a a + \sum_{a'\in \mathrm{A}} b_{a'}a' := \sum_{a\in \mathrm{A}} c_a + b_a a$$

and multiplication is given by

$$\left(\sum_{a\in \mathrm{A}} c_a a\right)\left(\sum_{a'\in \mathrm{A}} b_{a'}a'\right) = \sum_a \sum_{a'} c_a b_{a'}(a+a').$$

Prove that $k[\mathrm{A}]$ *is naturally a Hopf algebra. Write the corresponding group scheme as* $\mathbb{G}^{\mathrm{A}}$ *(note that* $\mathbb{G}^{\mathbb{Z}} = \mathbb{G}_m$*).*

(2) *Prove that we have an equivalence of categories*

$$\mathrm{Fun}(\mathrm{A}, \mathrm{Vect}_k) \simeq \mathrm{Rep}(\mathbb{G}^{\mathrm{A}}).$$

We note that the diagram A^{ds} *only depends on the order of* A*; what will distinguish them is the additive structure which will be reflected on the tensor structure of* $\mathrm{Fun}(\mathrm{A}^{\mathrm{ds}}, \mathrm{Vect}_k)$*, an idea that we will elaborate upon.*

EXERCISE 1.2.3 (Representation of $\mathbb{G}_a$). *We will work through the representation of* $\mathbb{G}_a$.

(1) *observe that a representation of* $\mathbb{G}_a$ *gives rise to a map*

$$\rho : \mathrm{V} \to \mathrm{V} \otimes k[\mathrm{T}];$$

hence we write $\rho(v) = \sum_{i\geqslant 0} \rho_i(v) \otimes t^i$*. This means we have maps*

$$\rho_i : \mathrm{V} \to \mathrm{V} \qquad i \geqslant 0.$$

Use the condition on the coaction to deduce that: $\rho_0 = \mathrm{id}$ *and that*

$$\rho_i\rho_j = \binom{i+j}{i}\rho_{i+j}.$$

(2) *In characteristic zero, prove that* $\rho_n = \frac{1}{n!}\rho_1^n$.

(3) Prove that, in characteristic zero, the category of $\mathbb{G}_a$-representation is the same thing as the category of locally nilpotent endomorphisms*: k-vector spaces* V *equipped with an endomorphism* T *such that for any $v \in \mathrm{V}$ $\mathrm{T}^{\circ k} v = 0$ for $k \gg 0$.*

2. Lie algebras

Arguably the next simplest algebraic group we can think of is SL_2. The representation theory of SL_2 is actually quite simple, but to describe it we do need more theory. What follows is the infinitesimal description of algebraic groups, a theme that we have alluded to throughout the lectures. In particular, we describe the key structure that's available to us.

Definition 2.0.1. A **Lie algebra** over a field k is a k-vector space $\mathfrak{g}$ equipped with a k-bilinear map

$$[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket such that

(1) $[x,x] = 0$ for all $x \in \mathfrak{g}$;
(2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all elements $x, y, z \in \mathfrak{g}$.

A **morphism** of Lie algebras is a k-linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\varphi([x,y]) = [\varphi(x), \varphi(y)].$$

We write Lie_k for the category of Lie algebras over k. A **sub-Lie algebra** of $\mathfrak{g}$ is a sub-vector space $\mathfrak{h} \subset \mathfrak{g}$ closed under the Lie bracket. A Lie algebra is **finite dimensional** if its underlying vector space is finite dimensional.

There are a couple of lemmas about Lie algebras which are standard and left to the reader to verify; they tell us how move brackets within a bracket:

Lemma 2.0.2. *Let $\mathfrak{g}$ be a Lie algebra with bracket $[-,-]$. Then:*

(1) $[-,-]$ is skew-symmetric: that is $[x,y] = -[y,x]$;
(2) for all $x, y, z \in \mathfrak{g}$, we have that $[x,[y,z]] - [y,[x,z]] = [[x,y],z]$.

The next two examples are key:

Example 2.0.3. If A is an associative k-algebra, we set

$$[a,b] = ab - ba$$

to get a Lie algebra. A key example for us will be the vector space $\mathrm{End}_k(\mathrm{V})$ of endomorphisms of a k-vector space V, which is an associative k-algebra under function composition. We write the Lie algebra as $\mathfrak{gl}(\mathrm{V})$.

Example 2.0.4. A more interesting example arises from the theory of **derivations**. If A is a commutative k-algebra, and M is an A-module then a **k-derivation valued in** M is a k-linear map

$$\mathrm{D} : \mathrm{A} \to \mathrm{M}$$

which satisfies Liebniz's rule:

$$\mathrm{D}(fg) = g\mathrm{D}(f) + f\mathrm{D}(g).$$

We write $\mathrm{Der}_k(\mathrm{A}, \mathrm{M})$ to be the set of k-linear derivations. We can add derivations pointwise to get a derivation and multiply by a scalar to get a derivation, hence $\mathrm{Der}_k(\mathrm{A}, \mathrm{M})$ is naturally a k-vector space.

Of particular interest for us is when $\mathrm{M} = \mathrm{A}$. Since a derivation is, in particular, a k-linear endomorphism of A we can compose them. However, be warned, that composition of a derivation is *not* a derivation. What one can check easily that the commutator:

$$[\mathrm{D}, \mathrm{E}] = \mathrm{D} \circ \mathrm{E} - \mathrm{E} \circ \mathrm{D}$$

is a derivation. In particular, we learn that $\mathrm{Der}_k(\mathrm{A}, \mathrm{A})$ is, in fact, a Lie algebra.

We want to study representation theory of algebraic groups using the representation theory of Lie algebras; the definition of the latter is:

DEFINITION 2.0.5. Let V be a (finite-dimensional) vector space, then a **(finite-dimensional) representation** of a Lie algebra $\mathfrak{g}$, valued in V, is a Lie algebra homomorphism

$$\mathfrak{g} \to \mathfrak{gl}(\mathrm{V}).$$

EXAMPLE 2.0.6 (Adjoint representation). The following is a key example, it practically determines the Lie bracket. Since $\mathfrak{g}$ is itself a vector space we can define the following representation:

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathrm{V}) \qquad x \mapsto [x, y].$$

The claim here is that

LEMMA 2.0.7. *The map* ad *is a Lie algebra homomorphism.*

We note that $\mathfrak{gl}(\mathrm{V})$ has its own Lie bracket irrespective of the original one from V.

PROOF. Let $x, y \in \mathfrak{g}$, then

$$\mathrm{ad}([x, y])(z) = [[x, y], z] = [x, [y, z]] = [y, [x, z]];$$

because of Lemma 2.0.2. On the other hand:

$$[\mathrm{ad}(x), \mathrm{ad}(y)](z) = ([x, -] \circ [y, -] - [y, -] \circ [x, -])(z) = [x, [y, z]] - [y, [x, z]].$$

□

This representation is called the **adjoint representation**.

2.1. Digression: Lie algebra and Lie groups. We now discuss the manifold version of what we are discussing, though the reader should be warned that the discussion will be informal. So let G be a Lie group, how does $\mathrm{T}_e\mathrm{G}$ look like? Well there are two ways to describe $\mathrm{T}_x\mathrm{M}$ for any manifold M;

(1) a tangent vector is an *derivation at x*: $\mathbb{R}$-linear maps

$$\mathrm{D}_x : \mathrm{C}^\infty(\mathrm{M}) \to \mathbb{R};$$

such that $\mathrm{D}_x(fg) = f(x)\mathrm{D}(g) + g(x)\mathrm{D}(f)$. This gives rise to the idea that things like $\frac{\partial}{\partial x_i}|_x$ is an element of the tangent space;

(2) a tangent vector is an equivalence class of a curve germ at x, that is to say a smooth map $\gamma : (-\epsilon, \epsilon) \to \mathrm{M}$ such that $\gamma(0) = x$ which are identified up to first order at zero.

The way to translate between them is "*take the derivative at x*." Therefore, if we take the derivation perspective, we can produce a curve germ. One of the main points of Lie theory is that, we can *extend* the curve germ to all of $\mathbb{R}$ using the group action; let's call the extension of γ by $\widetilde{\gamma}$. This gives rise to a rather surprising map called the *exponential*

$$\exp : \mathrm{T}_e\mathrm{G} \to \mathrm{G} \qquad \gamma \mapsto \widetilde{\gamma}(1).$$

Indeed, if $\mathrm{G} = \mathrm{S}^1$, then one can prove that $\mathrm{T}_e\mathrm{G} = \mathbb{R}$ and we get the real exponential $\mathbb{R} \to \mathrm{S}^1$ which is a local diffeomorphism.

Now, another new thing about the groupiness of the situation is that $\mathrm{T}_e\mathrm{G}$ has the structure of a Lie algebra. Actually Lie algebras abound everywhere in manifold theory. If M is a manifold then there are two ways of thinking about vector fields on M:

(1) the most slick definition is as a section of the tangent bundle $s : \mathrm{M} \to \mathrm{TM}$. That is to say: we pick for each $x \in \mathrm{M}$ a derivation at x, D_x and the assignment $s : x \mapsto \mathrm{D}_x$ is continuous in an appropriate sense;

(2) a more algebraic perspective is that it is a "global derivation": a derivation

$$\mathrm{D} : \mathrm{C}^\infty(\mathrm{M}) \to \mathrm{C}^\infty(\mathrm{M}).$$

Indeed, from a definition as in 1, we can produce the derivation that sends

$$f \mapsto \mathrm{D}(f)(x) = \mathrm{D}_x(f).$$

Now, by Example 2.0.4, we see that $\mathrm{C}^\infty(\mathrm{M})$ is a Lie algebra. This is, however, a huge Lie algebra which is probably not finite dimensional; it is related to the Lie group of self-diffeomorphisms of the manifold. If X is a Lie group, however, there is a smaller vector space which also inherits the Lie bracket. It is based on the following lemma:

Lemma 2.1.1. *There is an isomorphism between* $\mathrm{T}_e\mathrm{G}$ *and left-invariant vector fields. Furthermore,* $\mathrm{T}_e\mathrm{G}$ *is closed under the Lie bracket of vector fields.*

This Lie algebra is usually denoted by $\mathfrak{g}$; of course the underlying vector space is just $\mathrm{T}_e\mathrm{G}$. This leads to, again, two perspectives on what Lie algebras are in this analytic context:

(1) it is the Lie algebra of left-invariant vector fields on G;
(2) the Lie group G acts on G by conjugation; this is usually called the **adjoint action**. This action preserves the identity, hence for each $g \in \mathrm{G}$ we get a $\mathbb{R}$-linear map by taking derivatives:

$$\mathrm{Ad}(g)^* : \mathrm{T}_e\mathrm{G} \to \mathrm{T}_e\mathrm{G};$$

this gives us a map of Lie groups

$$\mathrm{G} \to \mathrm{GL}(\mathrm{T}_e\mathrm{G}).$$

Looking at the tangent space at the identity again we get a map

$$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

which one can prove is given by

$$x \mapsto [x, -].$$

The point of this digression is to motivate the definition of the "Lie algebras functor" which will expand on the second perspective above.

2.2. Algebraic groups and their Lie algebras. From now on we write ϵ for a variable that squares to zero: $\epsilon^2 = 0$. We will now expand on the relationship between algebraic groups and Lie algebras. From Exercise **??**, we learn how to define the tangent space at a k-point of a scheme. Indeed if X is a k-scheme, then the fact that $\mathrm{Hom}_{\mathrm{Sch}_k}(k[\epsilon], \mathrm{X})$ is described via the data of a k-point of X and an element in the tangent space:

$$\mathrm{T}_x\mathrm{X} = \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

can be reformulated as saying that we have the following pullback square (of sets):

$$\begin{tikzcd}
\mathrm{T}_x\mathrm{X} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Sch}_k}(k[\epsilon], \mathrm{X}) \arrow[d] \\
\{*\} \arrow[r, "x"] & \mathrm{Hom}_{\mathrm{Sch}_k}(k, \mathrm{X});
\end{tikzcd} \tag{2.2.1}$$

and further observing that $\mathrm{T}_x\mathrm{X}$ has a natural structure of a k-vector space. We remark that this is the analog of the curve germ story that we reviewed above.

Working with a group G rather than an arbitrary scheme, the tangent space of G at the identity can be described as the *kernel*:

$$\mathrm{T}_e\mathrm{G} = \ker\left(\mathrm{G}(k[\epsilon]) \to \mathrm{G}(k)\right).$$

Again it is a priori just a group, but it has the structure of k-vector space. Let us expand out the pullback (2.2.1) in the case of X = G to obtain some intuition about how the tangent space of a group looks like.

An element of $\mathrm{T}_e\mathrm{G}$ then corresponds to a morphism of k-algebras

$$\varphi : \mathcal{O}(\mathrm{G}) \to k[\epsilon]$$

with the condition that the composite

$$\mathcal{O}(\mathrm{G}) \xrightarrow{\varphi} k[\epsilon] \xrightarrow{\epsilon\mapsto 0} k$$

is the augmentation (unfortunately also written as ϵ). Let

$$\mathfrak{m}_e = \mathrm{I}_\mathrm{G}$$

be the kernel of augmentation, then $\varphi(\mathrm{I}_\mathrm{G}) \subset \epsilon$, hence φ uniquely factors through a map

$$\overline{\varphi} : \mathcal{O}(\mathrm{G})/\mathrm{I}_\mathrm{G}^2 \to k[\epsilon];$$

and we may write

$$\mathcal{O}(\mathrm{G})/\mathrm{I}_\mathrm{G}^2 \cong k \oplus \mathrm{I}_\mathrm{G}/\mathrm{I}_\mathrm{G}^2$$

and $\overline{\varphi}$ can be written as

$$(f, g) \mapsto f + \mathrm{D}(g)\epsilon$$

where $\mathrm{D}(g)$ is unique. This defines a map (which turns out to be an isomorphism)

$$\mathrm{T}_e(\mathrm{G}) \xrightarrow{\cong} \mathrm{Hom}_k(\mathrm{I}_\mathrm{G}/\mathrm{I}_\mathrm{G}^2(= \mathfrak{m}_e/\mathfrak{m}_e^2), k).$$

Remark 2.2.2. Most naturally, D is really a k-derivation $\mathcal{O}(\mathrm{G}) \to k$. Indeed, if we write $\varphi(f) = \epsilon(f) + \mathrm{D}(f)\epsilon$ then the fact that φ is homomorphism unpacks to the condition that $\mathrm{D} : \mathcal{O}(\mathrm{G}) \to k$ is a derivative. With this perspective the above isomorphism extends to another isomorphism:

$$\mathrm{Der}_k(\mathcal{O}(\mathrm{G}), k) \cong \mathrm{Hom}_k(\mathrm{I}_\mathrm{G}/\mathrm{I}_\mathrm{G}^2(= \mathfrak{m}_e/\mathfrak{m}_e^2), k).$$

We refer to Exercise 2.2.7 for a further elabration on this viewpoint.

To fix ideas we define

Definition 2.2.3. The **Lie algebra** of G is defined to be the k-vector space

$$\mathrm{Lie}(\mathrm{G}) := \mathrm{Hom}_k(\mathrm{I}_\mathrm{G}/\mathrm{I}_\mathrm{G}^2, k).$$

Exercise 2.2.4. *An element $\alpha \in k$ acts on $k[\epsilon]$ by sending ϵ to $\alpha\epsilon$, whence an endomorphism of* $\mathrm{T}_e\mathrm{G}$*. Prove that this is the same as scalar multiplication under the identification of* $\mathrm{T}_e\mathrm{G}$ *with* $\mathrm{Hom}_k(\mathrm{I}_\mathrm{G}/\mathrm{I}_\mathrm{G}^2, k)$*.*

We have yet to explain where the bracket is; once we have done that we write $\mathfrak{g}$ for Lie(G) with the bracket.

Example 2.2.5. An exercise above asks to compute $\mathrm{Lie}(\mathrm{SL}_n)$. Well $\mathrm{Lie}(\mathrm{GL}_n)$ is easier is much easier; it is isomorphic to

$$\mathrm{M}_n(k) \cong \{\mathrm{I}_n + \mathrm{X}\epsilon : \mathrm{X} \in \mathrm{M}_n(k)\}(\subset \mathrm{GL}(k[\epsilon]).$$

We can regard this as an isomorphism between

$$\mathrm{Hom}_k(\mathfrak{m}_e/\mathfrak{m}_2, k) \xrightarrow{\cong} \mathrm{T}_e\mathrm{G} \qquad \mathrm{X} \mapsto e^{\epsilon\mathrm{X}} = \mathrm{I} + \epsilon\mathrm{X};$$

inspired by the picture in differential topology of exponential being a local diffeomorphism between the tangent space at the identity and the group. The Lie algebra structure we should endow is of course the one given by the commutator

$$[\mathrm{X}, \mathrm{Y}] = \mathrm{XY} - \mathrm{YX}.$$

Remark 2.2.6. More generally, let us write

$$e^{\epsilon\mathrm{X}} \in \mathrm{T}_e(\mathrm{G}) \subset \mathrm{GL}(k[\epsilon])$$

for the image of $\mathrm{X} \in \mathrm{Lie}(\mathrm{G})$ under the isomorphism $\mathrm{Lie}(\mathrm{G}) \cong \mathrm{T}_e\mathrm{G}$. The point to remember here is that expressions like $1 + \epsilon\mathrm{X}$ "lies in the physical group" whereas X itself is an element of the k-linear vector space Lie(G).

Exercise 2.2.7. *We say that a k-derivation $\mathrm{D} : \mathcal{O}(\mathrm{G}) \to \mathcal{O}(\mathrm{G})$ is left invariant if the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{O}(\mathrm{G}) \arrow[r, "\mathrm{D}"] \arrow[d, "\Delta"] & \mathcal{O}(\mathrm{G}) \arrow[d] \\
\mathcal{O}(\mathrm{G}) \otimes \mathcal{O}(\mathrm{G}) \arrow[r, "\mathrm{id}\otimes\mathrm{D}"] & \mathcal{O}(\mathrm{G}).
\end{tikzcd}$$

Write $\mathrm{Der}_k(\mathcal{O}(\mathrm{G}), \mathcal{O}(\mathrm{G}))^\ell$ for the k-vector space of left invariant derivations. Prove:

(1) $\mathrm{Der}_k(\mathcal{O}(\mathrm{G}), \mathcal{O}(\mathrm{G}))^\ell$ is a sub-lie algebra of $\mathrm{Der}_k(\mathcal{O}(\mathrm{G}), \mathcal{O}(\mathrm{G}))$;

(2) prove that the map of k-vector spaces:

$$\mathrm{Der}_k(\mathcal{O}(\mathrm{G}), \mathcal{O}(\mathrm{G})) \to \mathrm{Der}_k(\mathcal{O}(\mathrm{G}), k) \qquad \mathrm{D} \mapsto \epsilon \circ \mathrm{D}$$

induces an isomorphism between $\mathrm{Der}_k(\mathcal{O}(\mathrm{G}), k)$ and $\mathrm{Der}_k(\mathcal{O}(\mathrm{G}), \mathcal{O}(\mathrm{G}))^\ell$.

2.3. Construction of the Lie Bracket.

We will prove the following theorem:

Theorem 2.3.1. *There is unique functor*

$$\mathrm{Lie} : \mathrm{AffGrp}_k \to \mathrm{Lie}_k$$

characterized by:

(1) $\mathrm{Lie(G)}$ is $\mathrm{Hom}_k(\mathrm{I_G}/\mathrm{I}_\mathrm{G}^2, k)$ as a k-vector space;

(2) the bracket on $\mathrm{Lie}(\mathrm{GL}_n) \cong \mathrm{M}_n(k)$ is given by

$$[\mathrm{X}, \mathrm{Y}] = \mathrm{XY} - \mathrm{YX}.$$

If we think of Lie(G) with its Lie bracket we write it as $\mathfrak{g}$. The first step is to construct an action of G on Lie(G).

Construction 2.3.2. We wish to construct, in algebraic geometry, the action of G on Lie(G), i.e., a morphism of algebraic groups

$$\mathrm{G} \to \mathrm{GL}(\mathrm{Lie(G)}).$$

First, let us understand Lie(G) better; we note that $\mathrm{T}_e\mathrm{G}$ extends as a functor via the exact sequence:

$$1 \to \mathrm{T}_e(\mathrm{G})(\mathrm{R}) \to \mathrm{G}(\mathrm{R}[\epsilon]) \to \mathrm{G(R)} \to 1.$$

On the other hand, Lie(G) is a k-vector space and we can extend it as a functor by just taking

$$\mathrm{R} \mapsto \mathrm{Lie(G)} \otimes \mathrm{R};$$

these two extensions are isomorphic as functors:

$$\mathrm{T}_e(\mathrm{G})(\mathrm{R}) \cong \mathrm{Hom_R}(\mathrm{I_G}/\mathrm{I}_\mathrm{G}^2 \otimes_k \mathrm{R}, \mathrm{R}) \cong \mathrm{Hom}(\mathrm{I_G}/\mathrm{I}_\mathrm{G}^2, k) \otimes_k \mathrm{R} \cong \mathrm{Lie(G)} \otimes \mathrm{R};$$

this is an R-linear extension of the correspondence we have already explained.

Now, by group theory, $\mathrm{G}(\mathrm{R}[\epsilon])$ acts via inner automorphisms (in other words, via conjugation) on $\mathrm{T}_e(\mathrm{G})(\mathrm{R})$. We note that G(R), sitting canonically as a subgroup of $\mathrm{G}(\mathrm{R}[\epsilon])$, also acts on $\mathrm{T}_e\mathrm{G}(\mathrm{R})$.

Exercise 2.3.3 (Linear action). *Prove that the action of G(R) on $\mathrm{T}_e\mathrm{G}(\mathrm{R})$ is R-linear (hint: extend Exercise 2.2.4).*

All in all, this builds a map

$$\mathrm{Ad} : \mathrm{G} \to \mathrm{GL}(\mathrm{T}_e\mathrm{G}).$$

Definition 2.3.4. The **adjoint representation** is the induced map on taking Lie (equivalently T_e) on Ad.

$$\mathrm{ad} : \mathrm{Lie(G)} \to \mathrm{End}(\mathrm{Lie(G)}) \cong \mathrm{Lie}(\mathrm{GL}(\mathrm{T}_e(\mathrm{G})).$$

The **Lie bracket** on Lie(G) is the pairing $[-,-]$ on Lie(G) given by

$$[\mathrm{X}, \mathrm{Y}] = \mathrm{ad}(\mathrm{X})(\mathrm{Y}).$$

We need one last lemma, left as an easy exercise in diagram chasing:

LEMMA 2.3.5. *The functor*

$$\mathrm{AffGrp}_k \to \mathrm{Vect}_k \qquad \mathrm{G} \mapsto \mathrm{T}_e\mathrm{G}$$

converts inclusions of (pointwise) injections of groups to injections of vector spaces.

PROOF OF THEOREM 2.3.1. We first observe that, by Lemma 2.3.5, the faithful representation $\mathrm{G} \to \mathrm{GL(V)}$ from Theorem 1.1.7 induces an injection of the underlying vector spaces of Lie(G)'s. Hence, requiring that the bracket on $\mathrm{Lie}(\mathrm{GL}_n)$ be the commutator pins down the values of the brackets on all other groups. This gives uniqueness of the functor.

We next observe that the bracket as defined in Definition 2.3.4 on GL_n, i.e., the adjoint action is indeed the "naively defined" commutator on n by n matrices. Let us try to compute the adjoint action (the recipe for computing Lie algebra action from an algebraic group action will be discussed shortly). We have to get somewhat organized: first and foremost, the action of GL_n on $\mathrm{Lie}(\mathrm{GL}_n) = \mathrm{M}_n$ can be described as follows: for every ring R, write elements of the latter as $1 + \epsilon \mathrm{X}$ where $\mathrm{X} \in \mathrm{M}_n(\mathrm{R})$; then the action of $g \in \mathrm{G(R)}$ on $\mathrm{M}_n(\mathrm{R})$ is given by conjugation, which makes sense because:

$$g(1+\epsilon\mathrm{X})g^{-1} = gg^{-1} + \epsilon g\mathrm{X}g^{-1} = \mathrm{id} + \epsilon g \mathrm{X} g^{-1}.$$

Indeed, the action of g on $\mathrm{M}_n(\mathrm{R})$ is by usual conjugation.

Now we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{gl}_n \arrow[r, "\mathrm{ad}"] \arrow[d] & \mathfrak{gl}_{\mathrm{T}_e\mathrm{G}} \arrow[d] \\
\mathrm{GL}_n(k[\epsilon]) \arrow[r, "\mathrm{Ad}_{k[\epsilon]}"] & \mathrm{GL}(\mathrm{T}_e\mathrm{G})(k[\epsilon]) = \mathrm{GL}(\mathrm{T}_e\mathrm{G}[\epsilon]);
\end{tikzcd}$$

and we wish to compute the top horizontal map. Going down and right sends a matrix X to $1 + \epsilon\mathrm{X}$ and, subsequently, to an operator that acts on $\mathrm{T}_e\mathrm{G}[\epsilon] = \mathrm{M}_n(k[\epsilon])$ as follows:

$$1 + \epsilon\mathrm{X} \cdot_{\mathrm{act}} (\mathrm{A} + \epsilon\mathrm{B}) = (1+\epsilon\mathrm{X})(\mathrm{A}+\epsilon\mathrm{B})(1+\epsilon\mathrm{X})^{-1} = (1+\epsilon\mathrm{X})(\mathrm{A}+\epsilon\mathrm{B})(1-\epsilon\mathrm{X});$$

where we have used that $\epsilon^2 = 0$ and the fact that the Ad-action, as observed in the previous paragraph, is simply conjugation. Now expanding this, we get

$$\mathrm{A} + \epsilon\mathrm{B} + \epsilon(\mathrm{XA} - \mathrm{AX}).$$

So, how do we think of this as an element of $\mathfrak{gl}_{\mathrm{T}_e\mathrm{G}}$? Remember that this is supposed to be $\mathrm{End}(\mathrm{T}_e\mathrm{G})$; under this identification an endomorphism T is supposed to act on $\mathrm{T}_e\mathrm{G}[\epsilon]$ via

$$(\mathrm{id} + \mathrm{T}\epsilon)(\mathrm{A} + \epsilon\mathrm{B}) = \mathrm{A} + \epsilon\mathrm{B} + \epsilon\mathrm{T}(\mathrm{A}).$$

So the endomorphism we get is just obtained by "extracting $\mathrm{XA} - \mathrm{XA}$", i.e., we get the commutator!

□

2.4. Computing the Lie algebra representation from an algebraic action. This section is about elaborating on the above idea involved in the proof of Theorem 2.3.1, it is not supposed to be deep by any means but it can get a bit confusing. Overall the point here is that we are trying to take derivatives algebraically.

Suppose that we are given an algebraic representation $\rho : \mathrm{G} \to \mathrm{GL(V)}$; how do we compute $\mathfrak{g} \to \mathfrak{gl}(\mathrm{V})$? The point here is to always use the commutative diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{Lie}(\rho)"] \arrow[d] & \mathfrak{gl}_{\mathrm{V}} \arrow[d] \\
\mathrm{G}(k[\epsilon]) \arrow[r, "\rho"] & \mathrm{GL}(\mathrm{V}[\epsilon]).
\end{tikzcd}$$

We take an element $\mathrm{X} \in \mathfrak{g}$ and send it to $1 + \epsilon\mathrm{X} \in \mathrm{G}(k[\epsilon])$. We then send it via ρ to an automorphism

$$\rho(1 + \epsilon\mathrm{X}) : \mathrm{V}[\epsilon] \to \mathrm{V}[\epsilon].$$

So one usually evaluates an expression like:

$$(1 + \epsilon\mathrm{X}) \cdot_{\mathrm{act}} v.$$

To extract the corresponding endomorphism of V one reverses the process: by commutativity of the above diagram the resulting automorphism of $\mathrm{V}[\epsilon]$ must be of the form $\mathrm{id} + \epsilon\mathrm{Y}$ where Y is an endomorphism of V and the action is by

$$(\mathrm{id} + \epsilon\mathrm{Y})(v + \epsilon w) = v + \epsilon w + \epsilon\mathrm{Y}v.$$

From this point of view: $\mathrm{Y} = \mathrm{Lie}(\rho)(\mathrm{X})$.

Example 2.4.1. We know that $\mathbb{G}_m$-representations are the same thing as graded vector spaces. Suppose that V is homogeneous of degree n, this means that V is one-dimensional with the $\mathbb{G}_m$ action given by

$$\lambda \cdot_{\mathrm{act}} v = \lambda^n v;$$

the reader is encouraged to write carefully this as a map $\rho_n : \mathbb{G}_m \to \mathrm{GL}_1$. Doing the procedure above: we know that $\mathrm{T}_e\mathbb{G}_m = k$ itself; write $\mathrm{X} \in k$. The automorphism

$$\rho_n(1 + \epsilon\mathrm{X}) : k[\epsilon] \to k[\epsilon]$$

is given by:

$$(1 + \epsilon\mathrm{X})^n\,(a + \epsilon b) \qquad \mathrm{X}, a, b \in k.$$

Using the binomial coefficients we see that the coefficient in front of ϵa is exactly n. Hence the endomorphism above is given by $a \mapsto na$.

Using the same idea, we can prove:

Exercise 2.4.2. *Consider the inclusion of $\mathbb{G}_m \to \mathrm{SL}_2$ via*

$$\mathbb{G}_m = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \lambda \in \mathrm{R}^{\times}$$

Prove the corresponding inclusion $\mathfrak{gl}_1 \to \mathfrak{sl}_2$ is given by

$$\mathfrak{gl}_1 = \begin{pmatrix} \mathrm{X} & 0 \\ 0 & -\mathrm{X} \end{pmatrix} \qquad \mathrm{X} \in k.$$

2.5. Representation theory of SL_2. Now we want to give another "hands-on" illustration of a category of representations. This will motivate the abstract machinery that will appear very soon in the notes and demonstrate the power of Lie algebras. By Theorem 2.3.1, we see that any G-representation $\mathrm{G} \to \mathrm{GL}(\mathrm{V})$ defines a Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(\mathrm{V})$; the V's here are the same V. Hence we have a functor

$$\mathrm{Rep}(\mathrm{G}) \to \mathrm{Rep}(\mathfrak{g}). \tag{2.5.1}$$

We will soon see that this functor is, in fact, an equivalence in large generality.

Now we just concentrate on $\mathrm{G} = \mathrm{SL}_2$. Its Lie algebra was seen to be the 2×2-traceless matrices and its Lie bracket is inherited from $\mathrm{M}_2(k)$.

Exercise 2.5.2. *Verify that $\mathfrak{sl}_2$ is 3-dimensional. Let*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Verify the identities

$$[e, f] = h \qquad [h, e] = 2e \qquad [h, f] = -2f.$$

We want to understand irreducible representations of $\mathfrak{sl}_2$; this means:

Definition 2.5.3. A representation of $\mathfrak{g}$ (resp. G) $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathrm{V})$ (resp. $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$) if there is no nontrivial (trivial means either $\{0\}$ or V itself) $\mathfrak{g}$-stable subspace (G-stable subspace) of V (resp. V).

Exercise 2.5.4. *Prove that the functor above preserves the category of irreducible representations.*

Remember that being $\mathfrak{g}$-stable means $\mathfrak{g}\mathrm{W} \subset \mathrm{W}$ for a subspace $\mathrm{V} \subset \mathrm{W}$; the same goes for the algebraic group side. So let us take an irreducible G-representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ which we assume to be irreducible; the corresponding $\mathfrak{g}$-representation is also irreducible, which we denoted by $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathrm{V})$.

Now, we make our first move: restrict along $\mathbb{G}_m \hookrightarrow \mathrm{SL}_2$ where the embedding is the one given in Exercise 2.4.2. In this case, the datum of a $\mathbb{G}_m$-action is the same as decomposing V as $\mathrm{V} \cong \bigoplus \mathrm{V}(j)$; we note that there are only finitely many i's that appear. But by Exercise 2.4.2, this means that the action of h on V decomposes it into these $\mathrm{V}(i)$'s as follows: an vector $v \in \mathrm{V}(i)$ if and only if $hv = jv$. In this case we say that v is **weight** j and that the space $\mathrm{V}(j)$ is the j-th **weight space**.

Returning to SL_2 or, rather, $\mathfrak{sl}_2$ we get, by Exercise (2.5.2), that

$$[h, e] = 2e.$$

Thus if v is of weight j we get that

$$hev_j = (i+2)ev_j;$$

hence e sends

$$e : \mathrm{V}(j) \to \mathrm{V}(j+2).$$

On the other hand, again by Exercise (2.5.2) we have that

$$hfv_j = (j-2)fv_j$$

and so f does

$$f : \mathrm{V}(j) \to \mathrm{V}(j-2).$$

But of course, ev_j could as well be zero. So we make the following definition:

Definition 2.5.5. A **highest weight space** of V is defined to be $\mathrm{V}(j)$ such that $\mathrm{V}(j) \neq 0$ but $\mathrm{V}(j+2)$ is zero. An element in the highest weight space is called a **highest weight vector**. A **lowest weight space** of V is defined to be $\mathrm{V}(j)$ such that $\mathrm{V}(j) \neq 0$ but $\mathrm{V}(j-2)$ is zero. An element in the lowest weight space is caleled the **lowest weight vector**.

At this point, we can illustrate the picture as follows:

$$\begin{tikzcd}
0 \arrow[r, bend left, "e"] & \mathrm{V}(\mathrm{lowest}) \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & \mathrm{V}(\mathrm{lowest}+2) \arrow[l, bend left, "f"] & \cdots & \mathrm{V}(\mathrm{highest}-2) \arrow[r, bend left, "e"] & \mathrm{V}(\mathrm{highest}) \arrow[l, bend left, "f"] & 0
\end{tikzcd}$$

There is something arbitrary about what we mean by "lowest" or "highest" though; this ultimately depends on the fact that we have chosen the way that $\mathbb{G}_m$ embeds inside SL_2. The following concept will appear more later on

Definition 2.5.6. The **normalizer** (resp. centralizer) of a subgroup H of G is the algebraic group defined by

$$\mathrm{N}_{\mathrm{G}}(\mathrm{H})(\mathrm{R}) = \{g \in \mathrm{G}(\mathrm{R}) : g\mathrm{H}(\mathrm{R})g^{-1} = \mathrm{H}(\mathrm{R})\}$$

$$\mathrm{C}_{\mathrm{G}}(\mathrm{H})(\mathrm{R}) = \{g \in \mathrm{G}(\mathrm{R}) : gh = hg \, \forall h \in \mathrm{H}(\mathrm{R}') \ \mathrm{R}' \text{ is an R-algebra}\}.$$

The **Weyl group of** SL_2 is the quotient

$$\mathrm{W}(\mathrm{SL}_2, \mathbb{G}_m) = \mathrm{N}_{\mathrm{SL}_2}(\mathbb{G}_m)/\mathrm{C}_{\mathrm{SL}_2}(\mathbb{G}_m).$$

Note that the centralizer of $\mathbb{G}_m$ in SL_2 is itself.

As in abstract group theory, one considers the Weyl group because it is the group of automorphisms of H which extends to inner automorphisms of G. In a sense, it is a very natural group that encodes the symmetries of how H lies inside G. In any case, we have not yet learned how to take quotient, but the answer turns out to be $\mathbb{Z}/2$.

EXERCISE 2.5.7. *Prove that* $\mathrm{W}(\mathrm{SL}_2, \mathbb{G}_m)(k) \cong \mathbb{Z}/2$ *whose nontrivial element by the matrix:*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, we have the action of $\mathrm{W}(\mathrm{SL}_2, \mathbb{G}_m)$ on $\mathbb{G}_m$ which intertwines the two possible ways that $\mathbb{G}_m$ sits inside SL_2. Taking derivatives, i.e., on the level of Lie algebras, we then have an action of $\mathrm{W}(\mathrm{SL}_2, \mathbb{G}_m)$ on $k \cdot h = \mathfrak{gl}_1$ inside $\mathfrak{sl}_2$. More precisely, we have an action of

$$\sigma \cdot_{\mathrm{act}} \begin{pmatrix} \mathrm{X} & 0 \\ 0 & -\mathrm{X} \end{pmatrix}.$$

How do we compute this action? Well on the level of algebraic groups, the action is by conjugation which one can compute

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)^{-1} = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \\ . & \end{pmatrix}$$

Whence,

$$\sigma \cdot_{\mathrm{act}} \begin{pmatrix} \mathrm{X} & 0 \\ 0 & -\mathrm{X} \end{pmatrix} = \begin{pmatrix} -\mathrm{X} & 0 \\ 0 & \mathrm{X} \end{pmatrix}.$$

How does this help us? Well we have used the original embedding $\mathbb{G}_m \hookrightarrow \mathrm{SL}_2$ to decompose V into its graded pieces. But we could also have used the "negative" of the above embedding: the Weyl group are automorphisms of $\mathbb{G}_m$ which extends to an inner automorphism of SL_2 so we can swap the original embedding with the other one. Of course, in practice, we are making the computation:

$$\sigma h \mathrm{V}(j) = \mathrm{V}(-j),$$

which probably does not need the language of the Weyl group to produce (we only need an appropriate involution on the torus). But the language of the Weyl group will be important later on. Anyway, we have learned more about our picture by exploiting this symmetry:

LEMMA 2.5.8. *Suppose that n is the highest weight vector, then $-n$ is the lowest weight vector.*

And so our picture looks like:

$$\begin{tikzcd}
0 \arrow[r, bend left, "e"] & \mathrm{V}(-\mathrm{highest}) \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & \mathrm{V}(\mathrm{lowest}+2) \arrow[l, bend left, "f"] & \cdots & \mathrm{V}(\mathrm{highest}-2) \arrow[r, bend left, "e"] & \mathrm{V}(\mathrm{highest}) \arrow[l, bend left, "f"] & 0.
\end{tikzcd}$$

So our pictures look like:

$$\begin{tikzcd}
\mathrm{V}(-1) \arrow[r, bend left, "e"] & \mathrm{V}(1) \arrow[l, bend left, "f"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{V}(-2) \arrow[r, bend left, "e"] & \mathrm{V}(0) \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & \mathrm{V}(2) \arrow[l, bend left, "f"] & .
\end{tikzcd}$$

We have yet to use the irreducibility of V though. The following uses characteristic zero:

Lemma 2.5.9. *Let k be a field of characteristic zero. For any $\mathfrak{g}$-representation* V. *Let $v := v_n \neq 0 \in \mathrm{V}(n)$. Take*

$$v, fv, \cdots, f^k v, \cdots, f^n v \qquad 0 \leqslant k \leqslant n.$$

Then the above vector are linearly independent and

$$\mathrm{Span}\{v, \cdots, f^n v\} \subset \mathrm{V}$$

is stable under e, f, h. In particular if V *is irreducible then:*

$$\mathrm{Span}\{v, \cdots f^n v\} = \mathrm{V}$$

Proof. One wants to prove Linear independence by claiming that 1) these are all h-eigenvectors with distinct eigenvalues as discussed above, 2) none of the $f^k v$'s is actually zero. Indeed, if we know 2), then 1) is clear because we are in characteristic zero and there is no way that the eigenvalues that appear are the same. We will postpone 2) in to latter paragraph. Instead we claim that these vectors are invariant under the action of f, h and e.

Indeed, h-invariance is clear as these are h-eigenvectors. The f-invariance of the vectors follow from the fact that $f^{n+1}v = 0$ since the symmetry argument above tells us that f^n must be the lowest weight vector. We now compute the action of e on these various vectors. Now $ev = 0$. We claim that:

Exercise 2.5.10. *For all $0 \leqslant k \leqslant n$ we have that*

$$e(f^k v) = k(n-k+1) f^{k-1} v.$$

Prove this by induction.

One of the things that this formula also does is prove linear independence: indeed $e^k f^k v$ is a nonzeromultiple of v, hence $f^k v$ could not have been zero. The proof is now complete.

□

At this point, things are quite satisfying: we know how these representations look like in the sense that we can compute their dimension, write down a basis if we want to and, in general, we just have a feel about how the structure of $\mathfrak{sl}_2$ acts on V. We can then ask ourselves: how do we model these representations? In other words, what are they concretely?

Definition 2.5.11. Let V, W be G-representations (with maps ρ_{V} and ρ_{W} respectively), then its **tensor product** is the representation with underlying vector space $\mathrm{V} \otimes_k \mathrm{W}$ and the action given by

$$\mathrm{G}(\mathrm{R}) \to \mathrm{GL}(\mathrm{V} \otimes_k \mathrm{W} \otimes_k \mathrm{R}) \qquad \rho_{\mathrm{V}}(g) \otimes \rho_{\mathrm{W}}(g)$$

In terms of comodules, suppose that we have

$$\rho_{\mathrm{V}} : \mathcal{O}(\mathrm{G}) \to \mathrm{V} \otimes \mathcal{O}(\mathrm{G}) \qquad \rho_{\mathrm{W}} : \mathcal{O}(\mathrm{G}) \to \mathrm{W} \otimes \mathcal{O}(\mathrm{G})$$

then the comodule structure is given by

$$\mathcal{O}(\mathrm{G}) \xrightarrow{\rho_{\mathrm{V}} \otimes \rho_{\mathrm{W}}} (\mathrm{V} \otimes \mathcal{O}(\mathrm{G})) \otimes (\mathrm{W} \otimes \mathcal{O}(\mathrm{G})) \cong (\mathrm{V} \otimes \mathrm{W}) \otimes (\mathcal{O}(\mathrm{G}) \otimes \mathcal{O}(\mathrm{G})) \xrightarrow{\mathrm{id} \otimes m} (\mathrm{V} \otimes \mathrm{W}) \otimes \mathcal{O}(\mathrm{G}).$$

We will soon assemble the tensor product above into a structure called a symmetric monoidal category. For now, we are interested in the following construction:

Exercise 2.5.12. *Let k be a field of characteristic zero. We have the standard representation $k^{\oplus 2}$ of SL_2; we can take its symmetric algebra*

$$\mathrm{Sym}^*(k^{\times 2}) = k \oplus k^{\oplus 2} \oplus ((k^{\oplus 2})^{\otimes_2})_{\Sigma_2} \oplus \cdots ((k^{\oplus 2})^{\otimes n})_{\Sigma_n} \cdots.$$

Endow each homogeneous component with a natural SL_2-action. Prove that such the n-th component is isomorphic to the unique irreducible representation of $\mathfrak{sl}_2$ with highest weight of weight n.

We have an isomorphism

$$k[x,y] \cong \mathrm{Sym}^*(k^{\times 2})$$

and an identification of n-th homogeneous polynomial with the degree n homogeneous piece of the symmetric algebra. Compute the action of SL_2 *and* $\mathfrak{sl}_2$ *on these homogeneous polynomials.*

Exercise 2.5.13. *Let* G *be an algebraic group and let* V *be a representation of* G *(not assumed to be over a field of characteristic 0). By definition,* $\mathrm{Sym}^2\mathrm{V}$ *is* $(\mathrm{V}^{\otimes 2})_{\Sigma_2}$. *One can also consider* $\Gamma_2\mathrm{V}$, *which by definition is* $(\mathrm{V}^{\otimes 2})^{\Sigma_2}$.

1. *Show that* $\Gamma_2\mathrm{V}$ *can be given the structure of a subrepresentation of* $\mathrm{V}^{\otimes 2}$.
2. *If* V *is finite-dimensional, show that* $\Gamma_2(\mathrm{V}^\vee) \cong (\mathrm{Sym}^2\mathrm{V})^\vee$.
3. *In characteristic 0, show that the composition* $\Gamma_2\mathrm{V} \to \mathrm{V}^{\otimes 2} \to \mathrm{Sym}^2\mathrm{V}$ *is an isomorphism.*

Exercise 2.5.14. *The representations of* SL_2 *look quite different in positive characteristic. Let's see a simple example of this. Consider* SL_2/k, *where k is a field of characteristic 2. Let* V_{std} *be the standard representation of* SL_2.

1. *Show that* $\mathrm{Sym}^2\mathrm{V}_{std}$ *isn't an irreducible representation of* SL_2.
2. *Show that* $\mathrm{Sym}^2\mathrm{V}_{std}$ *isn't even semisimple.*
3. *Show that* $\mathrm{V}_{std} \cong \mathrm{V}_{std}^\vee$ *(in any characteristic). Use Exercise 2.5.13 to conclude that* $\Gamma_2\mathrm{V}_{std} \ncong \mathrm{Sym}^2\mathrm{V}_{std}$ *over a field of characteristic 2.*

3. Categorical Preliminaries

We begin by gathering some preliminaries from category theory.

3.1. Tensor categories. We recall the notion of a symmetric monoidal category (also called a tensor category in the context of representation theory); sometimes these are also called tensor categories in the literature. Let $\mathcal{C}$ be a category and suppose that we have a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

An **associativity constrain** for $\mathcal{C}$ is a natural isomorphism

$$\varphi_{\mathrm{X,Y,Z}} : \mathrm{X} \otimes (\mathrm{Y} \otimes \mathrm{Z}) \xrightarrow{\cong} (\mathrm{X} \otimes \mathrm{Y}) \otimes \mathrm{Z}$$

satisfying the **pentagon axiom**:

$$\begin{tikzcd}
& (\mathrm{X} \otimes \mathrm{Y}) \otimes (\mathrm{Z} \otimes \mathrm{W}) \arrow[dr, "\varphi"] & \\
((\mathrm{X} \otimes \mathrm{Y}) \otimes \mathrm{Z}) \otimes \mathrm{W} \arrow[ur, "\varphi"] \arrow[d, "\varphi \otimes \mathrm{id}"'] & & \mathrm{X} \otimes (\mathrm{Y} \otimes (\mathrm{Z} \otimes \mathrm{W})) \\
(\mathrm{X} \otimes (\mathrm{Y} \otimes \mathrm{Z})) \otimes \mathrm{W} \arrow[rr, "\varphi"'] & & \mathrm{X} \otimes ((\mathrm{Y} \otimes \mathrm{Z}) \otimes \mathrm{W}) \arrow[u, "\mathrm{id} \otimes \varphi"']
\end{tikzcd}$$

A **commutativity constrain** for $\mathcal{C}$ is a natural isomorphism

$$\psi_{\mathrm{X,Y}} : \mathrm{X} \otimes \mathrm{Y} \xrightarrow{\cong} \mathrm{Y} \times \mathrm{X}$$

such that

$$\psi_{\mathrm{Y,X}} \circ \psi_{\mathrm{X,Y}} = \mathrm{id}.$$

We say that the commutativity and the associativity constrains are **compatible** if they satisfy the **hexagon axiom**:

$$\begin{tikzcd}
& \mathrm{X} \otimes (\mathrm{Y} \otimes \mathrm{Z}) \arrow[r, "\psi"] & (\mathrm{Y} \otimes \mathrm{Z}) \otimes \mathrm{X} \arrow[dr, "\varphi"] & \\
(\mathrm{X} \otimes \mathrm{Y}) \otimes \mathrm{Z} \arrow[ur, "\varphi"] \arrow[dr, "\psi \otimes \mathrm{id}"'] & & & \mathrm{Y} \otimes (\mathrm{Z} \otimes \mathrm{X}) \\
& (\mathrm{Y} \otimes \mathrm{X}) \otimes \mathrm{Z} \arrow[r, "\varphi"'] & \mathrm{Y} \otimes (\mathrm{X} \otimes \mathrm{Z}) \arrow[ur, "\mathrm{id} \otimes \psi"'] &
\end{tikzcd}$$

A **unit object** is an object $\mathbf{1} \in \mathcal{C}$ equipped with an isomorphism $u : \mathbf{1} \xrightarrow{\cong} \mathbf{1} \otimes \mathbf{1}$ such that the functor

$$\mathrm{X} \mapsto \mathbf{1} \otimes \mathrm{X}$$

is an equivalence of categories. A **symmetric monoidal category** is the data of $(\mathcal{C}, \otimes, \varphi, \psi, \mathbf{1}, u)$. One can prove rather easily that the identity object also satisfy natural isomorphisms $\mathrm{X} \simeq \mathrm{X} \otimes \mathbf{1}$ and is unique up to a unique isomorphism.

Example 3.1.1. If $\mathcal{C}$ has finite products and a terminal object $*$ (which is also known as the empty limit), then $(\mathcal{C}, \times, *)$ is an example of a symmetric monoidal category. The isomorphism datum in this case is *not* additional datum! It is determined by the universal property of the product within the category $\mathcal{C}$; in this case we say that the symmetric monoidal structure is the **Cartesian symmetric monoidal structure**.

Example 3.1.2. Let A be an abelian group. Then we have looked at A^{ds}, the category whose objects are elements of A with no non-trivial morphisms. We can define a symmetric monoidal structure on this via the addition law:

$$a \otimes b := a + b.$$

This is a rather degenerate case in which the associativity and commutativity constrains are given by identity maps. The reader is encouraged to check that this defines a symmetric monoidal structure on A^{ds}.

Example 3.1.3. Equipped with the usual $\otimes$-structure, the category $\mathrm{Mod}_{\mathrm{R}}$ of modules over a commutative ring R is a symmetric monoidal category.

Example 3.1.4. The category of graded k-vector spaces is a symmetric monoidal category with **Day convolution**:

$$(\mathrm{V}(*) \otimes \mathrm{W}(*))\,(n) := \bigoplus_{i+j=n} \mathrm{V}(i) \otimes \mathrm{W}(j).$$

The reader is invited to write down the associativity and commutativity constrains which are inherited from those from the category of k-vector spaces. In general,

$$\mathrm{Fun}(\mathrm{A}^{\mathrm{ds}}, \mathrm{Vect}_k)$$

admits a symmetric monoidal structure via Day convolution:

$$(\mathrm{V}(*) \otimes \mathrm{W}(*))\,(x) := \bigoplus_{a+b=x} \mathrm{V}(a) \otimes \mathrm{W}(b).$$

Exercise 3.1.5. *Prove that the above construction indeed furnishes a symmetric monoidal structure on* $\mathrm{Fun}(\mathrm{A}^{\mathrm{ds}}, \mathrm{Vect}_k)$.

Example 3.1.6 (Super vector spaces). An interesting class of examples in the representation theory literature are **super vector spaces**, these are vector spaces V equipped with a decomposition

$$\mathrm{V} = \mathrm{V}_0 \oplus \mathrm{V}_1$$

The tensor product is given also by Day convolution; explicitly:

$$(\mathrm{V} \otimes \mathrm{W})_0 = (\mathrm{V}_0 \otimes \mathrm{W}_0) \oplus (\mathrm{V}_1 \otimes \mathrm{W}_1);$$

$$(\mathrm{V} \otimes \mathrm{W})_1 = (\mathrm{V}_0 \otimes \mathrm{W}_1) \oplus (\mathrm{V}_1 \otimes \mathrm{W}_0).$$

The most interesting part of the symmetric monoidal structure on super vector spaces is the commutativity constrain determined by:

$$v \otimes w \mapsto (-1)^{ij} w \otimes v.$$

Sometimes we only a *monoidal* structure as opposed to a symmetric one: so this amounts to the tuple $(\mathcal{C}, \otimes, \varphi, \mathbf{1}, u)$; an example that will be interesting to us is:

Example 3.1.7. Let $\mathcal{C}$ be a category, then functor composition defines a monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ (which is not symmetric monoidal since there's no reason that $\mathrm{F} \circ \mathrm{G} = \mathrm{G} \circ \mathrm{F}$). The reader is left to verify this.

3.2. Closedness and rigidity. The "dual" notion to tensor products is the notion of hom/exponential objects.

Definition 3.2.1. Let $\mathrm{X}, \mathrm{Y} \in \mathcal{C}$. An **exponential of** X **by** Y is a morphism $e : \mathrm{X}^{\mathrm{Y}} \otimes \mathrm{Y} \to \mathrm{Y}$ characterized by the following universal property: for all $\mathrm{T} \in \mathcal{C}$, the map

$$\mathrm{Hom}(\mathrm{T}, \mathrm{X}^{\mathrm{Y}}) \to \mathrm{Hom}(\mathrm{T} \otimes \mathrm{Y}, \mathrm{X}^{\mathrm{Y}} \otimes \mathrm{Y}) \xrightarrow{e_*} \mathrm{Hom}(\mathrm{T} \otimes \mathrm{Y}, \mathrm{X})$$

is an isomorphism,.

The reader is highly encouraged to think about X^{Y} as $\underline{\mathrm{Hom}}(\mathrm{Y}, \mathrm{X})$, an "internal hom" and we will sometimes use the latter notation. The map e is a kind of evaluation map. We say that a tensor category $\mathcal{C}$ is **closed** if internal hom's exist for all pairs X and Y. This is an important condition for various manipulations that will follow.

Next, we will discuss the notion of dualizability. Most people understand dualizable objects as something like $\underline{\mathrm{Hom}}(\mathrm{X}, \mathbf{1})$; in the context of vector spaces this is the notion of the functional dual. The right notion of dualizability is a bit more elaborate.

Definition 3.2.2. We say that X is **dualizable** if there exists an object $\mathrm{X}^\vee$ and maps

$$\mathrm{coev} : \mathbf{1} \to \mathrm{X} \otimes \mathrm{X}^\vee \qquad \mathrm{ev} : \mathrm{X}^\vee \otimes \mathrm{X} \to \mathbf{1}$$

such that the composites

$$\mathrm{X} \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} (\mathrm{X} \otimes \mathrm{X}^\vee) \otimes \mathrm{X} \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} \mathrm{X}$$

and

$$\mathrm{X}^\vee \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} \mathrm{X}^\vee \otimes (\mathrm{X} \otimes \mathrm{X}^\vee) \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} \mathrm{X}^\vee$$

are equal to the identity. We call $(\mathrm{X}^\vee, \mathrm{ev}, \mathrm{coev})$ the **duality datum** for X.

Exercise 3.2.3. *The key example is of course $\mathcal{C} = \mathrm{Vect}_k$, the category of k-vector spaces. A vector space* V *is dualizable if and only if it is finite dimensional. The key point here (and the only thing you need to explain) is that why the fact that the the two composites in the definition above being the identity implies that* V *must be finite dimensional.*

Remark 3.2.4. In a monoidal category, we have an asymmetric notion of left and right dualizability. While we will not formulate this, in the example of the monoidal category of endofunctors, being left (right) dualizable means exactly that a functor F has a left (right) adjoint. The reader is encouraged to play around with this notion.

At first glance, it seems that dualizability is a structure (one needs to furnish the datum of the maps above) and not a property. This is not the case: being dualizable is equivalent to the following: for any other object T, we have an isomorphism

$$\mathrm{T} \otimes \mathrm{X}^{\vee} \simeq \underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{T}).$$

In particular, we always have the formula

$$\mathrm{X}^{\vee} \simeq \underline{\mathrm{Hom}}(\mathrm{X}, \mathbf{1}),$$

whenever X is dualizable.

Proposition 3.2.5. *Assume that* $\mathbf{C}$ *is a closed tensor category. Suppose that we are given a map* $e : \mathrm{D} \otimes \mathrm{X} \to k$*. Then* e *extends to a duality datum* (D, e, c) *if and only if for any object* T*, the map*

$$\mathrm{T} \otimes \mathrm{D} \otimes \mathrm{X} \xrightarrow{\mathrm{id} \otimes e} \mathrm{T}$$

exhibits $\mathrm{T} \otimes \mathrm{D}$ *as the exponential object* $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{T})$*: for any other object* T' *the map*

$$\mathrm{Hom}(\mathrm{T}', \mathrm{T} \otimes \mathrm{D}) \to \mathrm{Hom}(\mathrm{T}' \otimes \mathrm{X}, \mathrm{T} \otimes \mathrm{D} \otimes \mathrm{X}) \xrightarrow{\mathrm{id} \otimes e_*} \mathrm{Hom}(\mathrm{T}' \otimes \mathrm{X}, \mathrm{T})$$

is an isomorphism.

Proof. Assume that D extends to a duality datum and fix T. The inverse to the above isomorphism is furnished by coev:

$$\mathrm{Hom}(\mathrm{T}' \otimes \mathrm{X}, \mathrm{T}) \to \mathrm{Hom}(\mathrm{T}' \otimes \mathrm{X} \otimes \mathrm{D}, \mathrm{T} \otimes \mathrm{D}) \xrightarrow{\mathrm{coev}^*} \mathrm{Hom}(\mathrm{T}', \mathrm{T} \otimes \mathrm{D}),$$

the reader is encouraged to check this claim.

Assume that second condition is satisfied, then plugging in $\mathrm{T} = \mathrm{X}, \mathrm{T}' = \mathbf{1}$ we get

$$\mathrm{Hom}(\mathbf{1}, \mathrm{X} \otimes \mathrm{D}) \cong \mathrm{Hom}(\mathrm{X}, \mathrm{X})$$

and the coevaluation map is the image of the identity under the above isomorphism. ☐

Remark 3.2.6 (Traces). By Proposition 3.2.5, we have that $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{X}) \cong \mathrm{X}^{\vee} \otimes \mathrm{X}$ whenever X is dualizable; and so the evaluation map takes the form $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{X}) \to \mathbf{1}$. Applying $\mathrm{Hom}(\mathbf{1}, -)$ we get

$$\mathrm{Hom}(\mathrm{X}, \mathrm{X}) \to \mathrm{Hom}(\mathbf{1}, \mathbf{1});$$

which we call the **trace** map. In the case of vector spaces, this coincides with the usual trace. The **rank** of an object X is the element in $\mathrm{Hom}(\mathbf{1}, \mathbf{1})$ given by the image of $\mathrm{id} \in \mathrm{Hom}(\mathrm{X}, \mathrm{X})$.

Definition 3.2.7. We say that a tensor category $\mathcal{C}$ is **rigid** if all objects are dualizable.

3.3. Monads and comonads. This will be quite abstract, so let us begin with a motivating example. Consider the free and forgetful adjunction

$$\mathrm{F} : \mathrm{Set} \rightleftarrows \mathrm{Grp} : o.$$

Then we have the endofunctor $\mathbb{F} := o \circ \mathrm{F} = o\mathrm{F} : \mathrm{Set} \to \mathrm{Set}$. We wish to make precise the following idea: a group is the datum of an set with an "$\mathbb{F}$"-action. Indeed, note that if G is a group then $o\mathrm{G}$ is just the group G regarded as a set. In this case, we have a map

$$a : \mathbb{F}(o\mathrm{G}) = o\mathrm{F}o(\mathrm{G}) \xrightarrow{\mathrm{id} \circ \epsilon} o\mathrm{G}.$$

In coordinates: we have taken the free group on the underlying set of G, called $\mathbb{F}(o\mathrm{G})$ whose elements are words in $o\mathrm{G}$. Then, using the fact that G is a group, we can map it back into $o\mathrm{G}$, remembering all the relations the words in G has. In fact, the relations can be encoded in a similar way. We have two maps

$$\mathbb{F}(a), : \mathbb{F}\mathbb{F}(o\mathrm{G}) \to \mathbb{F}(o\mathrm{G});$$

The first

All of this is encoded in the theory of monads and comands which are pervasive throughout mathematics.

The format of the theory will look like this: $\mathbb{F}$ will have the structure of a **monad** which is the notion of an algebra in endofunctors. The functor o will factor through an appropriate category of $\mathbb{F}$-modules; this encodes the fact that we can map back $\mathbb{F}$ on $o\mathrm{G}$ back to $o\mathrm{G}$:

$$\begin{tikzcd}
\mathrm{Grp} \arrow[r, "o^{\mathrm{enh}}"] \arrow[d, "o"] & \mathrm{Mod}_{\mathbb{F}}(\mathrm{Set}) \arrow[dl] \\
\mathrm{Set} \arrow[u, bend left, "\mathrm{F}"] &
\end{tikzcd}$$

Here o^{enh} should be thought of as an "enhanced" version of o. One can then prove the following result:

Proposition 3.3.1. *We have an equivalence of categories* $\mathrm{Grp} \xrightarrow{o^{\mathrm{emh}}} \mathrm{Mod}_{\mathbb{F}}(\mathrm{Set})$.

This makes precise the ideal that groups are "modules over $\mathbb{F}$."

To set up the language of monads and comonads, we need to talk about abstract algebras and coalgebras:

Definition 3.3.2. Let $(\mathcal{C}, \otimes)$ be a monoidal category. Then an **algebra object** in $\mathcal{C}$ is an object A equipped with maps

$$m : \mathrm{A} \otimes \mathrm{A} \to \mathrm{A},$$

is the "multiplication",

$$\eta : \mathbf{1} \to \mathrm{A},$$

is the "unit." These are subject to the commutativity of the following diagrams:

(Associativity)

$$\begin{tikzcd}
\mathrm{A} \otimes \mathrm{A} \otimes \mathrm{A} \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] & \mathrm{A} \otimes \mathrm{A} \arrow[d, "m"] \\
\mathrm{A} \otimes \mathrm{A} \arrow[r, "m"] & \mathrm{A}.
\end{tikzcd}$$

(Identity)

$$\begin{tikzcd}
\mathbf{1} \otimes \mathrm{A} \arrow[r, "\eta \times \mathrm{id}"] \arrow[dr, "\simeq"'] & \mathrm{A} \otimes \mathrm{A} \arrow[d, "m"] & \mathrm{A} \otimes \mathbf{1} \arrow[l, "\mathrm{id} \times \eta"'] \arrow[dl, "\simeq"] \\
& \mathrm{A} &
\end{tikzcd}$$

Say that $\mathcal{C}$ is furthermore symmetric monoidal, we say that A is **commutative** if furthermore the diagram

(Commutativity)

$$\begin{tikzcd}
\mathrm{A} \otimes \mathrm{A} \arrow[rr, "\psi"] \arrow[dr, "m"'] & & \mathrm{A} \otimes \mathrm{A} \arrow[dl, "m"] \\
& \mathrm{A} &
\end{tikzcd},$$

commutes.

Given an associative algebra A, a **left A-module** is an object M equipped with an "action" map

$$a : \mathrm{A} \otimes \mathrm{M} \to \mathrm{M}$$

such that the diagram

(Associativitiy)

$$\begin{tikzcd}
\mathrm{A} \otimes \mathrm{A} \otimes \mathrm{M} \arrow[r, "\mathrm{id} \times a"] \arrow[d, "m \times \mathrm{id}"'] & \mathrm{A} \otimes \mathrm{M} \arrow[d, "m"] \\
\mathrm{A} \otimes \mathrm{M} \arrow[r, "a"] & \mathrm{M}.
\end{tikzcd}$$

(Identity)

$$\begin{tikzcd}
\mathbf{1} \otimes \mathrm{M} \arrow[r, "\eta \times \mathrm{id}"] \arrow[dr, "\simeq"'] & \mathrm{A} \otimes \mathrm{M} \arrow[d, "a"] \\
& \mathrm{M}
\end{tikzcd}$$

.

Remark 3.3.3. Taking opposites, the reader can formulate the notion of coalgebras, comodules etc. They are invited to spell this out.

We will actually mostly care about comonads, so let us spell out a special example of a coalgebra:

Definition 3.3.4. Let $\mathcal{C}$ be a category. A **comonad** is a functor

$$\mathrm{T} : \mathcal{C} \to \mathcal{C}$$

equipped with transformations

$$u : \mathrm{T} \to \mathrm{id};$$

$$m : \mathrm{T} \to \mathrm{T} \circ \mathrm{T},$$

such that the diagrams

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "m"] \arrow[d, "m"'] & \mathrm{T} \circ \mathrm{T} \arrow[d, "m \circ \mathrm{id}"] \\
\mathrm{T} \circ \mathrm{T} \arrow[r, "\mathrm{id} \circ m"] & \mathrm{T} \circ \mathrm{T} \circ \mathrm{T},
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "m"] \arrow[d, "m"'] & \mathrm{T} \circ \mathrm{T} \arrow[d, "u \circ \mathrm{id}"] \\
\mathrm{T} \circ \mathrm{T} \arrow[r, "\mathrm{id} \circ u"] & \mathrm{T}.
\end{tikzcd}$$

commute. In other words, it is a coalgebra object in $(\mathrm{Fun}(\mathcal{C}, \mathcal{C}), \circ)$.

We are mostly interested in comonads but one might be more familiar with the notion of a monad where we are given maps $\mathrm{id} \to \mathrm{T}$ and $\mathrm{T} \circ \mathrm{T} \to \mathrm{T}$; so T almost looks like a group object but instead of × we have ∘ and there is no notion of inverses.

Example 3.3.5. Our interest in comonads arise in the following manner: let $\mathcal{O}(\mathrm{G})$ be the ring of functions of a group, then we have an endofunctor

$$\mathrm{T}_{\mathrm{G}} := - \otimes \mathcal{O}(\mathrm{G}) : \mathrm{Vect}_k \to \mathrm{Vect}_k$$

From the fact that $\mathcal{O}(\mathrm{G})$ is a coalgebra, we see that the functor T_{G} is in fact a comonad; note that we ignore the multiplication structure in this definition.

Example 3.3.6. In fact, if A is an associative k-algebra, then $- \otimes \mathrm{A} : \mathrm{Vect}_k \to \mathrm{Vect}_k$ is a monad.

The most important source of comonads are:

Exercise 3.3.7. *Let* $\mathrm{L} : \mathcal{C} \to \mathcal{D}$ *be a functor with right adjoint* $\mathrm{R} : \mathcal{D} \to \mathcal{C}$. *Then we have the counit map*

$$\epsilon : \mathrm{LR} \to \mathrm{id}$$

and the map

$$m : \mathrm{LR} \simeq \mathrm{L}(\mathrm{id})\mathrm{R} \xrightarrow{\eta} \mathrm{LRLR}.$$

Verify that this defines a comonad structure on the endofunctor

$$\mathrm{LR} : \mathcal{D} \to \mathcal{D}.$$

Now, let us note the following: if we have an object of the form $\mathrm{LX} \in \mathcal{D}$ then the unit transformation $\eta : \mathrm{id} \to \mathrm{RL}$ induces a map

$$a : \mathrm{LX} \xrightarrow{\mathrm{id}\circ\eta} \mathrm{LRLX};$$

furthermore we have the following commutativity from properties of the unit transformation

$$\begin{tikzcd}
\mathrm{LX} \arrow[r, "a"] \arrow[d, "a"] & \mathrm{LRLX} \arrow[d, "m\circ\mathrm{id}"] \\
\mathrm{LRLX} \arrow[r, "\mathrm{id}\circ a"] & \mathrm{LRLRLX}.
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{id}\circ\mathrm{LX} & \mathrm{LRLX} \arrow[l, "\epsilon\circ\mathrm{id}"'] \arrow[r, "\mathrm{id}\circ\epsilon"] & \mathrm{LX}\circ\mathrm{id} \\
& \mathrm{LX} \arrow[ul, "="] \arrow[u, "a"] \arrow[ur, "="'] &
\end{tikzcd}$$

We are trying to do representation theory, so we are interested in actions.

Definition 3.3.8. Let $\mathrm{T} : \mathcal{C} \to \mathcal{C}$ be a comonad. Then a (left) **T-comodule** is a comodule for T. Spelling this out we have a functor:

$$\mathrm{M} : \mathcal{C} \to \mathcal{C},$$

a coaction map

$$\mathrm{M} \to \mathrm{T} \circ \mathrm{M},$$

satisfying the condition for M to be a comodule over T.

We have seen that $\mathrm{T} = \mathrm{LR}$ is a comonad. There is a natural way to produce T-comodules.

Lemma 3.3.9. *Let* $\mathrm{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{R}$ *be an adjunction. Then the functor* L *factors canonically as*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{L}^{\mathrm{enh}}"] \arrow[dr, "\mathrm{L}"] & \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D}) \arrow[d] \\
& \mathcal{D}.
\end{tikzcd}$$

The functor $\mathrm{L}^{\mathrm{enh}}$ *is called the* ***enhanced left adjoint****.*

Remark 3.3.10. Concretely, a T-comodule structure on an endofunctor M means that for each $\mathrm{X} \in \mathcal{C}$ we have maps $\mathrm{MX} \to \mathrm{TMX}$ such that certain diagrams commute.

Proof. The point is that we have a natural transformation

$$\mathrm{id} \circ \epsilon : \mathrm{L} \Rightarrow \mathrm{LRL} = \mathrm{TL},$$

which endows L with the comodule structure. The identities that ensure L is a comodule follows from the identities that the unit and counit of adjunctions satisfy. □

All of this it to set us up for one of the most important theorems in modern mathematics. It produces a condition for when the enhanced left adjoint induces an equivalence

$$\mathrm{L}^{\mathrm{enh}} : \mathcal{C} \xrightarrow{\simeq} \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D}).$$

3.4. forks, equalizers and split pairs. Before we state the Barr-Beck theorem, let us recall the formalism of equalizers; a **cofork** is a diagram

$$\mathrm{C} \overset{e}{\to} \mathrm{X} \rightrightarrows \mathrm{Y} \qquad \delta^0, \delta^1 : \mathrm{X} \to \mathrm{Y};$$

such that

$$\delta^0 e = \delta^1 e.$$

An equalizer of the maps δ_0 and δ_1 is the universal example of a cofork. A **cosplitting** of a cofork is the datum:

$$\mathrm{C} \overset{s}{\leftarrow} \mathrm{X} \overset{t}{\leftarrow} \mathrm{Y}$$

satisfying:

$$\delta^0 e = \delta^1 e \qquad se = \mathrm{id} \qquad t\delta^0 = \mathrm{id} \qquad es = \delta^1 t.$$

The key property of a splitting is that it affords an "explicit witness" for a cofork to be an equalizer diagram.

LEMMA 3.4.1. *If a cofork admits a splitting, then it is an equalizer diagram.*

Given a functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$, then a pair of parallel arrows $\mathrm{X} \rightrightarrows \mathrm{Y}$ in $\mathcal{C}$ is said to be a **F-cosplit** pair if $\mathrm{FX} \rightrightarrows \mathrm{FY}$ embeds into a split equalizer.

EXAMPLE 3.4.2.

Having went through the above digression we can formulate the Barr-Beck theorem:

THEOREM 3.4.3 (Barr-Beck). *Let* $\mathrm{L} : \mathcal{C} \to \mathcal{D}$ *be a functor with right adjoint* $\mathrm{R} : \mathcal{D} \to \mathcal{C}$. *Then the following are equivalent:*

(1) the enhanced functor $\mathrm{L}^{\mathrm{enh}} : \mathcal{C} \to \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D})$ *is an equivalence of categories;*

(2) L *is conservative and any* L*-cosplit pair in* $\mathcal{C}$ *admits an equalizer in* $\mathcal{C}$ *and is preserved by* L.

For references, we refer the reader to [**?**, IV.7]. The Barr-Beck theorem is very powerful; for our purposes we really only need the following version:

THEOREM 3.4.4 (Barr-Beck lite). *Let* $\mathrm{L} : \mathcal{C} \to \mathcal{D}$ *be a functor with right adjoint* $\mathrm{R} : \mathcal{D} \to \mathcal{C}$*, assume:*

(1) both $\mathcal{C}$ *and* $\mathcal{D}$ *admits equalizers;*

(2) L *preserves all equalizers;*

(3) L *is conservative.*

PROOF. By Lemma 3.3.9 we get a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{L}'"] \arrow[dr, "\mathrm{L}"'] & \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D}) \arrow[d] \\
 & \mathcal{D}.
\end{tikzcd}$$

As a first step, we construct a right adjoint $\mathrm{R} : \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D}) \to \mathcal{C}$. An object of $\mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D})$ is, in particular, an object $d \in \mathcal{D}$ equipped with the comodule structure

$$c : d \to \mathrm{LR}d.$$

To land in $\mathcal{C}$, we apply R to the diagram above to get a map $\mathrm{R}c : \mathrm{R}d \to \mathrm{RLR}d$. There is, however, a natural transformation $\mathrm{R} \to \mathrm{RLR}$ given by η_{R} and thus we have two maps

$$\eta_{\mathrm{R}c}, \mathrm{R}c : \mathrm{R}d \rightrightarrows \mathrm{RLR}d.$$

We then take the equalizer in $\mathcal{C}$ (which exists by hypothesis) and get an object $\mathrm{R}'d$ fitting as the equalizer:

$$\mathrm{R}'d \to \mathrm{R}d \rightrightarrows \mathrm{RLR}d.$$

The construction is functorial in $\mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D})$ and so defines a functor $\mathrm{R}' : \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D}) \to \mathcal{C}$.

We sketch a proof that R' is right adjoint to L'. Recall that an adjunction is determined by a pair of functors, R', L' and a candidate unit transformation $\mathrm{id} \to \mathrm{R}'\mathrm{L}'$ satisfying certain identities [**?**, Chapter IV, Theorem 2]. We give the candidate unit map: indeed, we want to map $c \in \mathcal{C}$ naturally to $\mathrm{R}'\mathrm{L}'c$. But now, we have an equalizer diagram (by construction above, setting $d = \mathrm{L}c$)

$$\mathrm{R}'\mathrm{L}'c \to \mathrm{RL}c \rightrightarrows \mathrm{RLRL}c;$$

where the two maps are

$$\mathrm{R}(\mathrm{L}\eta_c) \qquad \eta_{\mathrm{RL}c};$$

indeed the first map determines the RL coaction on $\mathrm{L}c$. The unit map $c \to \mathrm{RL}c$ is equalized under the two maps, whence we have a natural map $c \to \mathrm{R}'\mathrm{L}'c$. This is the candidate unit transformation and the reader is encouraged to check that it is indeed a unit.

Now, to prove equivalence, we observe that L' is conservative because L is. Therefore, by Lemma 3.4.5, it suffices to prove that R' is fully faithful, i.e., that the transformation $\mathrm{L}'\mathrm{R}' \to \mathrm{id}$ is an equivalence. We now take an object $d \in \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D})$ and compute $\mathrm{L}'\mathrm{R}'d$; indeed since L' is assumed to be equalizer-preserving, the resulting $\mathrm{L}'\mathrm{R}'d$ fits into the equalizer diagram (in $\mathcal{D}$):

$$\mathrm{L}'\mathrm{R}'d \to \mathrm{LR}d \rightrightarrows \mathrm{LRLR}d.$$

We will then be done if the following is proved: the diagram

$$d \to \mathrm{LR}d \rightrightarrows \mathrm{LRLR}d$$

is an equalizer diagram for any $d \in \mathrm{CoMod}_{\mathrm{LR}}(\mathcal{D})$ where the first map is the structure map of d as an LR-comodule. In fact, the diagram above admits a cosplitting:

$$d \xleftarrow{\epsilon} \mathrm{LR}d \xleftarrow{\mathrm{LR}\epsilon} \mathrm{LRLR}d,$$

which the reader is free to check. ☐

Lemma 3.4.5. *An adjoint pair* $\mathrm{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{G}$ *is an equivalence if and only if* F *(resp.* G*) is conservative and* G *(resp.* F*) is fully faithful.*

3.5. Ind-objects. To make effective use to the Barr-Beck theorems, we need to place ourselves in a situation where we have adjoints. Typically this requires some appeal to Freyd's adjoint functor theorem [**?**, V.7]. In the situation we will encounter, however, the adjoint will actually be quite explicit since we need only construct filtered colimits:

Definition 3.5.1. A diagram J is **filtered** if any two objects $j, j' \in \mathrm{J}$ admits an object k with maps $j \to k, j' \to k$ and any two parallel arrows $j \rightrightarrows j'$ completes to a fork.

As is usual we say that a category $\mathcal{C}$ **admits filtered colimits** if any diagram $\mathrm{J} \to \mathcal{C}$ where J is filtered admits a colimit in $\mathcal{C}$. We want to "freely adjoin" filtered colimits in $\mathcal{C}$, i.e,, objects of the form "$\mathrm{colim}_j \mathrm{X}_j$" subject to the requirement that existing objects are **compact**:

$$\mathrm{Hom}(\mathrm{C}, \text{“}\operatorname*{colim}_j \mathrm{X}_j\text{”}) \cong \operatorname*{colim}_j \mathrm{Hom}(\mathrm{C}, \mathrm{X}_j) \qquad \mathrm{C} \in \mathcal{C}.$$

Recall that in a category $\mathcal{C}$ with filtered colimits we say that an object C is compact if the canonical map

$$\operatorname*{colim}_{\mathrm{J}} \mathrm{Hom}(\mathrm{C}, \mathrm{X}_j) \to \mathrm{Hom}(\mathrm{C}, \mathrm{colim}\, \mathrm{X}_j)$$

for all filtered diagrams $j \mapsto \mathrm{X}_j$. This concept is yet another abstract formulation of being "finite dimensional."

Now, let $\mathcal{C}$ be an esssentially small category. Then there is procedure to formally adjoin filtered colimits and the resulting category is called the **ind-category** of $\mathcal{C}$ and denoted by $\mathrm{Ind}(\mathcal{C})$. Its objects are given by diagrams indexed by I where I is filtered, i.e., functors $\mathrm{I} \to \mathcal{C}$. We denote such diagrams by "$\mathrm{colim}_{i \in \mathrm{I}} \mathrm{X}_j$" and the Hom-sets are

$$\mathrm{Hom}(\text{“}\operatorname*{colim}_j \mathrm{Y}_j\text{”}, \text{“}\operatorname*{colim}_i \mathrm{X}_i\text{”}) = \operatorname*{colim}_i \lim_j \mathrm{Hom}(\mathrm{Y}_j, \mathrm{X}_i);$$

the order of the lim and the colim ensures the image of the functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ sending $c \in \mathcal{C}$ to the constant diagram at c lands in the subcategory of compact objects.

Proposition 3.5.2. *Let $\mathcal{C}$ be an essentially small category and let $\mathcal{D}$ be a category with filtered colimits, then the functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ induces a fully faithful functor*

$$\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where the domain consists of those functors that preserves filtered colimits.

Example 3.5.3. Let $\mathcal{C} = \mathrm{Vect}_k^{\mathrm{fd}}$ be the category of finite dimensional k-vector spaces. Then $\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{Vect}_k$, the category of all vector spaces. The idea here is that any vector space can indeed be written as a filtered colimit of finite dimensional ones (we just write it as a filtered union of its finite dimensional subspaces).

Now assume that $\mathcal{C}$ is an essentially small category and that $\mathcal{C}$ admits finite colimits (including the empty colimit, i.e., an initial object), then there is a concrete model for $\mathrm{Ind}(\mathcal{C})$. Denote by $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ the subcategory of those presheaves on $\mathcal{C}$ which converts finite colimits to finite limits (equivalently, takes finite limits in $\mathcal{C}^{\mathrm{op}}$ to finite limits). Observe that the Yoneda functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ factors through $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$; to see this one notes that $\mathrm{Hom}(-, \mathrm{X})$ does convert finite colimits to finite limits.

Proposition 3.5.4. *There is an equivalence of categories $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \simeq \mathrm{Ind}(\mathcal{C})$ under which the constant functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ identifies with the Yoneda functor.*

The basic idea behind Proposition 3.5.2 is that filtered colimits are exactly those colimits which commute with finite limits. This fits into a general pattern; for example the presheaf category on $\mathcal{C}$ has no restriction on the kind of colimits that a functor preserves and the resulting category adjoins all colimits, the usual universal property of the presheaf category.

The upshot here is that one should try to work with the more concrete model $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. For us it helps to compute explicit adjoints. Let $\mathcal{C}, \mathcal{D}$ be essentially small categories and suppose that $f : \mathcal{C} \to \mathcal{D}$ is a functor that preserves finite colimits. On the one hand, the universal property of Ind as in Proposition 3.5.2 furnishes the dotted arrows:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[r] & \mathrm{Ind}(\mathcal{D}) \\
\mathrm{Ind}(\mathcal{C}) \arrow[rru, dashed, "\mathrm{Ind}(f)"'] & &
\end{tikzcd} .$$

Concretely, $\mathrm{Ind}(f)$ takes a filtered diagram in $\mathcal{C}$, say “$\mathrm{colim}\, \mathrm{X}_j$” to the filtered diagram in $\mathcal{D}$ given by “$\mathrm{colim}\, f(\mathrm{X}_j)$”.

On the other hand we have the precomposition functor

$$f^* : \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \qquad \mathrm{F} \mapsto \mathrm{F} \circ f^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}} \to \mathrm{Set};$$

which factors through the ind-categories via the identification of Proposition 3.5.4.

Lemma 3.5.5. *We have an adjoint pair*

$$\mathrm{Ind}(f) : \mathrm{Ind}(\mathcal{C}) \rightleftarrows \mathrm{Ind}(\mathcal{D}) : f^*.$$

Proof. Let us sketch a proof; for an essentially small category $\mathcal{E}$ let us write $\mathrm{PSh}(\mathcal{E})$ for $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathrm{Set})$. Note that $\mathrm{Ind}(f)$ of a constant diagram c is just $f(c)$ again. Abusing notation, we regard a constant object via its Yoneda embedding. In this case we compute:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ind}(\mathcal{D})}(f(c), \mathrm{G}) &\cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(f(c), \mathrm{G}) \\
&= \mathrm{G}(f(\mathrm{C})) \\
&= (f^*\mathrm{G})(\mathrm{C}) \\
&\cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(c, f^*\mathrm{G}) \\
&\cong \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(c, f^*\mathrm{G}).
\end{aligned}$$

Now, we have a natural transformation

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{D})}(\mathrm{Ind}(f)(-), \mathrm{G}) \to \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(-, f^*\mathrm{G})$$

which is an isomorphism on the image of $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ and furthermore preserves filtered colimits, hence is an isomorphism on all of $\mathrm{Ind}(\mathcal{C})$. $\square$

The fact f^* is so explicit leads to the following result which is somewhat a surprising feature of a right adjoint:

LEMMA 3.5.6. *The functor f^* preserves all colimits.*

PROOF. It clearly preserves
Let $\mathrm{G} : \mathrm{I} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ be a diagram. We want to prove that $\mathrm{colim}_{\mathrm{I}} f^*\mathrm{G}_i \cong f^* \mathrm{colim}_{\mathrm{I}} \mathrm{G}_i$. First, notice that both $\square$

3.6. Applying the Barr-Beck theorem. Let us now see the Barr-Beck theorem in action

THEOREM 3.6.1. *Let $\mathcal{C}$ be an essentially small abelian k-linear category. Let $\omega : \mathcal{C} \to \mathrm{Vect}_k^{\mathrm{fd}}$ be an exact, conservative functor to finite dimensional vector spaces. Then:*

1. *we have an adjoint pair:*

$$\mathrm{Ind}(\omega) : \mathrm{Ind}(\mathcal{C}) \rightleftarrows \mathrm{Vect} : \mathrm{R},$$

furthermore the functor R *preserves colimits;*

2. *the adjoint pair above induces an equivalence of categories*

$$\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{CoMod}_{\mathrm{Ind}(\omega)\mathrm{R}}(\mathrm{Vect}_k);$$

3. *now, let us assume that $\mathcal{C}$ is a symmetric monoidal category and furthermore that ω is a strong symmetric monoidal functor then*

$$\mathrm{B} := \omega(\mathrm{R}(k)) \in \mathrm{Vect}_k$$

acquires the structure of a comonad;

4. *the equivalence from (2) induces an equivalence of categories*

$$\mathcal{C} \simeq \mathrm{CoMod}_{\mathrm{B}}(\mathrm{Vect}_k^{\mathrm{fd}})$$

4. The reconstruction theorem

We now have all the categorical ingredients to prove Tannaka duality. We recall the motivation from Fourier analysis at the beginning of this chapter. Any locally compact abelian group A has a Haar measure, with respect to this we have the space $\mathrm{L}^2(\mathrm{A})$ of square-integrable functions on A. As noted above, $\widehat{\mathrm{A}}$ is discrete and the natural notion of square-integrability leads to the "small ℓ^2-space" $\ell^2(\widehat{\mathrm{A}})$ if square-summable series. Pontrayagin duality asserts that an isomorphism

$$\mathrm{L}^2(\mathrm{A}) \cong \ell^2(\widehat{\mathrm{A}});$$

asserting that information about A is basically contained in its space of characters. Now, if G is not abelian, we see that $\widehat{\mathrm{G}}$ tends to be much smaller than G; indeed any group homomorphism $\mathrm{G} \to \mathrm{S}^1$ factors through the abelianlization, whence $\widehat{\mathrm{G}}$ only depends on G^{ab}. For example, if $\mathrm{G} = \Sigma_n$, the symmetric group on n-letters, then G^{ab} is $\mathbb{Z}/2$. The goal of the tannakian reconstruction theorem is to attempt to still describe G using its representation theory.

The main theorem of this chapter is

THEOREM 4.0.1. *Let $\mathcal{C}$ be a rigid, abelian k-linear tensor category such that* $\mathrm{End}(\mathbf{1}) = k$. *Let*

$$\omega : \mathcal{C} \to \mathrm{Vect}_k^{\mathrm{fd}}$$

be an exact, conservative tensor functor. Then

1. *the functor*
$$\mathrm{Aut}^{\otimes}(\omega) : \mathrm{CAlg}_k \to \mathrm{Grp} \qquad \mathrm{R} \mapsto \mathrm{Aut}^{\otimes}(\omega_{\mathrm{R}})$$
is represented by an algebraic group G*;*
2. *there is a canonical equivalence of tensor categories:*
$$\mathcal{C} \simeq \mathrm{Rep}(\mathrm{G})$$
over $\mathrm{Vect}_k^{\mathrm{fd}}$.

It is useful to introduce the following definition:

Definition 4.0.2. A rigid, abelian k-linear tensor category $\mathcal{C}$ such that $\mathrm{End}(\mathbf{1}) = k$ is called a **neutral Tannakian category (over k)** if it admits a conservative exact functor $\omega : \mathcal{C} \to \mathrm{Vect}_k$; the latter is called **fiber functor**.

Example 4.0.3. Let's examine briefly what happens when $\mathcal{C} = \mathrm{Rep}(\mathrm{G})$, the category of finite dimensional representation of G:

1. the tensor product of representation has, as underlying vector spaces, the tensor product of underlying vector spaces; this is exactly saying that the forgerful functor is symmetric monoidal.
2. the dual representation

4.1. Producing a monoid scheme. We have yet to produce the algebra structure on our coalgebra B from Theorem 3.6.1. An algebra structure on B is, in particular, a morphism $m : \mathrm{B} \otimes_k \mathrm{B} \to \mathrm{B}$. This lets us make the following maneuver: we can define a pairing:

$$\Phi^m : \mathrm{coMod}_{\mathrm{B}} \times \mathrm{coMod}_{\mathrm{B}} \to \mathrm{coMod}_{\mathrm{B}}$$

$$(\mathrm{X}, \mathrm{Y}) \mapsto (\mathrm{X} \otimes \mathrm{Y} \to (\mathrm{X} \otimes \mathrm{B}) \otimes (\mathrm{Y} \otimes \mathrm{B}) \cong (\mathrm{X} \otimes \mathrm{Y}) \otimes (\mathrm{B} \otimes \mathrm{B}) \to \mathrm{X} \otimes \mathrm{Y} \otimes \mathrm{B}.$$

Proposition 4.1.1. *Let* B *be a coalgebra.*

1. *the assignment*
$$m \mapsto \Phi^m$$
induces a one-to-one correspondence between maps $\mathrm{B} \otimes_k \mathrm{B} \to \mathrm{B}$ *and functors*
$$\mathrm{F} : \mathrm{coMod}_{\mathrm{B}} \times \mathrm{coMod}_{\mathrm{B}} \to \mathrm{coMod}_{\mathrm{B}}.$$
such that $\mathrm{F}(\mathrm{X}, \mathrm{Y}) = \mathrm{X} \otimes \mathrm{Y}$ *as vector spaces (i.e., after forgetting down).*
2. *the functor* F *defines a symmetric monoidal structure on* $\mathrm{coMod}_{\mathrm{B}}$ *such that the forgetful functor* $\mathrm{coMod}_{\mathrm{B}} \to \mathrm{Vect}_k$ *is symmetric monoidal if and only if* B *is a commutative* k*-algebra.*

Proof. □

4.2. The Tannaka dual group. We now need to be slightly more concrete so let us work "backwards." Let $\omega : \mathrm{Rep}(\mathrm{G}) \to \mathrm{Vect}_k$ be the forgetful functor which we want to examine more closely. Let us write $\mathrm{End}^{\otimes}(\omega)$ for the set of endomorphisms

$$\lambda : \omega \to \omega$$

commuting with tensor products. What this means concretely:

1. for each representation V, we have k-linear maps
$$\mathrm{V} \xrightarrow{\lambda_{\mathrm{V}}} \mathrm{V}$$
which are natural for morphisms of representations;
2. the morphism $\lambda_{\mathrm{V} \otimes \mathrm{W}}$ is equal to the morphism $\lambda_{\mathrm{V}} \otimes \lambda_{\mathrm{W}}$ and
3. $\lambda_{\mathrm{id}} = \mathbf{1}$.

In fact, for any k-algebra R, we can also look at $\mathrm{End}^{\otimes}(\omega_{\mathrm{R}})$ which admits the same description as above. The following result tells us that the functor

$$\underline{\mathrm{End}} : \mathrm{R} \mapsto \mathrm{End}^{\otimes}(\omega_{\mathrm{R}})$$

recovers G.

Theorem 4.2.1 (Baby Tannaka duality). *Let* G *be an algebraic group over* k *and fix a* k*-algebra* R*. Suppose that for every finite dimensional representation* V *of* G *we are given* R*-linear maps*

$$\lambda_{\mathrm{V}} : \mathrm{V}_{\mathrm{R}} \to \mathrm{V}_{\mathrm{R}}$$

such that

(1) $\lambda_{\mathrm{V}\otimes\mathrm{W}} = \lambda_{\mathrm{V}} \otimes \lambda_{\mathrm{W}}$*;*
(2) $\lambda_{\mathbf{1}} = \mathrm{id}$
(3) for any morphism of representations $f : \mathrm{V} \to \mathrm{W}$ *we have that* $\lambda_{\mathrm{W}} \circ f_{\mathrm{R}} = f_{\mathrm{R}} \circ \lambda\mathrm{V}$.

Then there exists a unique $g \in \mathrm{G}(\mathrm{R})$ *such that*

$$\lambda_{\mathrm{V}} = g \cdot_{\mathrm{act}} \cdot$$

Proof. ☐

Instead of End we can look at

$$\underline{\mathrm{Aut}} \subset \underline{\mathrm{End}}$$

the subfunctor where every morphism is invertible. The above result tells us that the above is in fact an isomorphism of functors.

Hence, if one believes in Tannaka duality, one should believe that G is constructed quite explictly as the $\otimes$-automorphism group of the functor ω.

4.3. Application I: Jordan decompositions.

Remark 4.3.1 (Jordan normal form).

4.4. Application II: reconstructing a group from its Lie algebra.

CHAPTER 4

Reductive groups

We now come to the heart of the course: the classification of reductive groups. We first need to introduce the different "flavors" of groups that can appear. The first flavor are called "unipotent groups" and these are, in some sense, really quite simple but exhibit pathologies we want to dismiss in having a good theory.

1. Unipotent groups

From linear algebra, a unipotent matrix is one of the form:

$$\begin{bmatrix} 1 & x_{12} & x_{13} & \dots & x_{1n} \\ 0 & 1 & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Here's something rather interesting about unipotent elements:

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}.$$

Strangely, matrix multiplication is converted to scalar *addition.* In fact the above presents the group $\mathbb{G}_a$ "multiplicatively." The notion of unipotent groups abstracts the above concept and gives an intrinsic characterization of matrices of the above form. There is actually already an abstract notion of what it means for an element to be unipotent, in an abstract ring.

Definition 1.0.1 (Unipotent element). An element $r \in \mathrm{R}$ is said to be **unipotent** if there exists an integer $\mathrm{N} \gg 0$ such that $(r-1)^{\mathrm{N}} = 0$.

In the ring $\mathrm{M}_n(k)$, matrices of the above form are precisely the unipotent ones. There is also a Tannakian-theoretic approach, based on the following result:

Exercise 1.0.2. *Let* $\mathrm{G} \subset \mathrm{GL}(\mathrm{V})$*, where* $\mathrm{V} \neq 0$*. Then if* G *acts via unipotent matrices, there exists an element* $v \in \mathrm{V}$ *such that* $\mathrm{G} \cdot v = v$*, i.e.,* $\mathrm{V}^{\mathrm{G}} \neq 0$

Let's see why this might be a problem for us. Suppose that V is a simple (in other words, irreducible) representation. Then we see that $\mathrm{V}^{\mathrm{G}} = \mathrm{V}$ necessarily since $\mathrm{V}^{\mathrm{G}} \neq 0$ and $\mathrm{V}^{\mathrm{G}} \subset \mathrm{V}$. Hence we must conclude that G acts trivially on any simple representation. This is very pathological. In fact, let us recall that the category of finite dimensional $\mathbb{G}_a$-representation is equivalent to the category of vector spaces equipped with a nilpotent endomorphisms; such a category is very far from being semisimple.

Remark 1.0.3 (Representation theory of finite groups). Let us recall the following result which is often stated as "Maschke's theorem" or the "averaging trick"

Theorem 1.0.4. *If* G *is a finite group and* $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ *is a finite dimensional representation of* G *over a field* K *which does not divide* $|\mathrm{G}|$*. Then if* $\mathrm{W} \subset \mathrm{V}$ *is a subrepresentation, we have a decomposition of* G*-representations*

$$\mathrm{V} \cong \mathrm{W} \oplus \mathrm{U},$$

where U *is a representation of* G*. Therefore, we can decompose any* G*-representation into a direct sum of irreducible representations.*

This, along with Schur's lemma, are two nice categorical properties of the representation of finite groups based on vector spaces over a field of characteristic zero: they are semisimple. The phenomenon of unipotence is exactly what obstructs this; if we are looking at representations of p-groups valued in $\mathbb{F}_p$ vector spaces then there are plenty of examples where representations cannot be decomposed into simples.

Hence we are on a mission to dismiss this phenomenon. We want a useful description of what it means for a group to be unipotent; in particular we want to have a definition of the "unipotent radical" of a group — the largest, *normal* subgroup of a group G which is unipotent. This is so that we can quotient this group out. First, let us begin with the obvious definition of U_n as an algebraic group.

Definition 1.0.5. The group U_n is defined as the functor

$$\mathrm{R} \mapsto \begin{bmatrix} 1 & * & & \dots & * \\ 0 & 1 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \subset \mathrm{GL}_n(\mathrm{R}).$$

Exercise 1.0.6. *Prove that* U_n *is a subgroup of* GL_n *which is, furthermore, smooth.*

A more intrinsic characterization is as follows:

Definition 1.0.7. An affine algebraic group over k is **unipotent** if every nonzero representation has a fixed vector.

There is also a reasonable notion for a representation to be unipotent. This contrasts against the notion of semisimple representations where we have a direct sum decomposition into simples, which are not trivial.

Definition 1.0.8. A representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ is **unipotent** if there exists a G-stable flag

$$\{0\} \subset \mathrm{V}_s \cdots \subset \mathrm{V}_1 \subset \mathrm{V}_0 = \mathrm{V}$$

such that each quotient $\mathrm{V}_j/\mathrm{V}_{j+1}$ admits a trivial G-action.

This also hints at a characterization of the Hopf algebra associated to C to be filtered with certain conditions:

Definition 1.0.9. A Hopf algebra A is **coconnected** if there is a filtration $\mathrm{C}_0 \subset \mathrm{C}_1 \subset \cdots \mathrm{A}$ where each C_i is a k-subspace such that

(1) $\mathrm{C}_0 = k$
(2) $\cup \mathrm{C}_r = \mathrm{A}$
(3) $\Delta(\mathrm{C}_r) \subset \sum_{i=0}^{r} \mathrm{C}_i \otimes \mathrm{C}_{r-i}$.

Example 1.0.10. Let us look at the Hopf algebra $\mathcal{O}(\mathrm{U}_n)$. Its underlying k-algebra is given by

$$k[\mathrm{X}_{ij} : i < j]$$

with the multiplication (induced by the one from $\mathcal{O}(\mathrm{G}_n) \to \mathcal{O}(\mathrm{U}_n)$ given by

$$\Delta(\mathrm{X}_{ij}) = \mathrm{X}_{ij} \otimes 1 + 1 \otimes \mathrm{X}_{ij} + \sum_{i<\ell<j} \mathrm{X}_{i\ell} \otimes \mathrm{X}_{\ell j}.$$

This k-algebra admits a filtration: we set

$$|\mathrm{X}_{ij}| = j - 1$$

and set $\mathcal{O}(\mathrm{U}_n)_r$ be the subspace of degree at most r polynomials. We check below that this indeed shows that $\mathcal{O}(\mathrm{U}_n)$ is a Hopf algebra.

We will now give various characterizations for an affine algebraic group to be unipotent. This will give us a sense of what kind of groups we are dismissing when we carry out the classification theory of reductive groups.

THEOREM 1.0.11. *Let* G *be affine algebraic group over a perfect field. Then the following are equivalent:*

1. G *is unipotent;*
2. *every finite dimensional representation of* G *is unipotent;*
3. *there is an* $\mathrm{N} \gg 0$ *and an embedding of groups*

$$\mathrm{G} \hookrightarrow \mathrm{U}_{\mathrm{N}}$$

4. $\mathcal{O}(\mathrm{G})$ *is coconnected.*

If k *is furthremore perfect and* G *is smooth, then this is equivalent to saying that any* $g \in \mathrm{G}(k)$ *is unipotent in terms of its multiplicative Jordan decomposition.*

PROOF. If G is unipotent, and V is a representation, we construct a G-stable flag by induction on dimension. Indeed, let v be the nonzero vector fixed by G, then V/v is a G-representation of one less dimension, and we can construct a G-stable flag by induction. On the other hand, the lowest member of the flag, V_s is a nonzero subspace which is fixed by G.

Now, assume 2). Then we can find a finite dimensional faithful representation of G by a previous result; it has a flag as in 2). Therefore G must be a subgroup of U_{N}. Now, assume 3); we note that quotients of coconnected Hopf algebras are coconneced: if $\mathrm{C} \to \mathrm{C}'$ is surjective then we set C'_r to be the image of C_r. It then suffices to prove that $\mathcal{O}(\mathrm{U}_n)$ is coconnected as we were trying to do in Example 1.0.10. It is easy to see that $\mathcal{O}(\mathrm{G})_0 = k, \cup \mathcal{O}(\mathrm{G})_r = \mathrm{A}$ and $\mathcal{O}(\mathrm{G})_i \mathcal{O}(\mathrm{G})_j \subset \mathcal{O}(\mathrm{G})_{i+j}$, i.e., we have a filtered k-algebra. We need only check the condition on the coproduct. For the X_{ij} we see that the formula for Δ in Example 1.0.10 gives us the result. In general, proceed by induction.

Lastly, assume 4) and we wish to prove that G is unipotent. Any comodule $\rho : \mathrm{V} \to \mathrm{V} \otimes \mathcal{O}(\mathrm{G})$ defines a filtration on V:

$$\mathrm{V}_r := \{v : \rho(v) \in \mathrm{V} \otimes \mathcal{O}(\mathrm{G})_r\}$$

Now, if V_0 contains a nonzero vector, then

$$\rho(v) \in \mathrm{V} \otimes k = \mathrm{V}$$

so that $\rho(v) = v'$; but $v = \epsilon\rho(v) = \epsilon(v' \otimes 1) = v'$ and hence $v = v'$ and is thus a fixed vector. Too finish the proof, we need to prove that $\mathrm{V}_r = 0$ means that $\mathrm{V}_{r+1} = 0$ so that if $\mathrm{V}_0 = 0$ then $\mathrm{V} = 0$. By construction $\rho(\mathrm{V}_{r+1}) \subset \mathrm{V} \otimes \mathcal{O}(\mathrm{G})_{r+1}$; by the condition on the coproduct we then have that

$$((\mathrm{id} \otimes \Delta) \circ \rho)(\mathrm{V}_{r+1}) \subset \mathrm{V} \otimes \sum_i \mathcal{O}(\mathrm{G})_i \otimes \mathcal{O}(\mathrm{G})_{r+1-i}$$

Hence $((\mathrm{id} \otimes \Delta) \circ \rho)(\mathrm{V}_{r+1})$ maps to zero under $\mathrm{V} \otimes \mathcal{O}(\mathrm{G}) \otimes \mathcal{O}(\mathrm{G}) \to \mathrm{V} \otimes \mathcal{O}(\mathrm{G})/\mathcal{O}(\mathrm{G})_r \otimes \mathcal{O}(\mathrm{G})/\mathcal{O}(\mathrm{G})_r$. Since, by assumption $\mathrm{V}_r = 0$, we have that

$$\rho : \mathrm{V} \to \mathrm{V} \otimes \mathcal{O}(\mathrm{G})/\mathcal{O}(\mathrm{G})_r$$

is injective because there is no loss in information in quotienting out $\mathcal{O}(\mathrm{G})_r$; similarly for $(\rho \otimes \mathrm{id}) \circ \rho$. But this is equal to $(\mathrm{id} \otimes \Delta) \circ \rho$, hence $\mathrm{V}_{r+1} = 0$.

Now we work over a perfect field. If G is unipotent, it is clear that $\mathrm{G}(k)$ consists of unipotent elements by embedding it into a faithful representation (by Jordan decomposition, to check unipotence of an element it suffices to check that the matrix of an embedding into a faithful representation is unipotent). Now, if $\mathrm{G}(k)$ is unipotent then we can find an embedding $\mathrm{G}(k) \subset \mathrm{U}_{\mathrm{N}}(k)$; this is embedding of closed subgroups. Since G is smooth, then $\mathrm{G}(k)$ is dense in G so we get an embedding $\mathrm{G} \subset \mathrm{U}_{\mathrm{N}}$.

□

Remark 1.0.12. Be warned that some authors use the last part of Theorem 1.0.11 to define a unipotent group so that, automatically, G must be smooth.

Before we prove this result, let us observe certain corollaries:

Corollary 1.0.13. *Unipotence is stable under subgroups, extensions and quotients.*

Proof. Let $\mathrm{U} \to \mathrm{Q}$ be a quotient, then any Q-representation is also a U-representation hence Q is unipotent if U is. Now, let $1 \to \mathrm{N} \to \mathrm{U} \to \mathrm{Q} \to 1$ be an extension with N, Q unipotent. Since N is a normal subgroup, V^{N} is U-stable, and the U-action factors through Q. Therefore, if V is nonzero, then $\mathrm{V}^{\mathrm{U}} = (\mathrm{V}^{\mathrm{N}})^{\mathrm{Q}}$ is nonzero. To see that unipotence is stable under subgroups, we use 3 of Theorem 1.0.11. □

Corollary 1.0.14. *A group* G *is unipotent if and only if for any extension* k'/k, $\mathrm{G}_{k'}$ *is unipotent if and only if for an algebraically closed field* $\overline{k}/k$, $\mathrm{G}_{\overline{k}}$ *is unipotent.*

Proof. This follows from the following observation: $(\mathrm{V} \otimes k')^{\mathrm{G}_k} = \mathrm{V}^{\mathrm{G}} \otimes k'$. Hence if $(\mathrm{V} \otimes k')^{\mathrm{G}_k} \neq 0$, then V^{G} cannot be zero. □

Definition 1.0.15. By Corollary 1.0.13, there exists a normal subgroup $\mathrm{R}_u(\mathrm{G})$ of G which is the largest, connected, unipotent normal subgroup of G. This subgroup is called the **unipotent radical**.

Corollary 1.0.16. *If* G *is smooth over a perfect field. Then* G *is unipotent if and only if* $\mathrm{G}(\overline{k})$ *is unipotent.*

Exercise 1.0.17. *Let* k'/k *be a Galois extension. Prove that* $\mathrm{R}_u(\mathrm{G}) \otimes_k k' \cong \mathrm{R}_u(\mathrm{G}_{k'})$.

Example 1.0.18. Consider the following flag of subgroups:

$$e = \{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\} \subset \{\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\} \subset \{\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}\} \subset \mathrm{U}_3;$$

which we write as

$$e = \mathrm{U}_3^{(3)} \subset \mathrm{U}_3^{(2)} \subset \mathrm{U}_3^{(1)} \subset \mathrm{U}_3$$

Observe:

(1) the inclusions above are inclusions of normal subgroups
(2) each subgroup that appears is normal in U_3
(3) each graded piece $\mathrm{U}_3^{(j)}/\mathrm{U}_3^{(j+1)}$ is normal in $\mathrm{U}_3/\mathrm{U}_3^{(j+1)}$.

Such a series is called a **cental normal series**. Note that the graded pieces are all isomorphic to $\mathbb{G}_a$, generalizing the phenomenon we saw at the beginning of this section. In fact:

Proposition 1.0.19. *Every unipotent group admits a central normal series whose quotients are isomorphic to subgroups of* $\mathbb{G}_a$.

This proposition then characterizes unipotent groups in terms of having nontrivial maps to $\mathbb{G}_a$.

Corollary 1.0.20. G *is unipotent if and only if every nontrivial subgroup of* G *admits a nontrivial map to* $\mathbb{G}_a$.

2. Groups of multiplicative type

The "opposite" of a unipotent group is called a group of **multiplicative type**. To begin with let us recall that group $\mathbb{G}_{\mathrm{A}}$ from Exercise 1.2.2; here A is an abelian group and $\mathbb{G}_{\mathrm{A}} = \operatorname{Spec} k[\mathrm{A}]$ where $k[\mathrm{A}]$ is the group algebra of an abstract abelian group A.

Exercise 2.0.1. *Let the functor*

$$\mathrm{D(A)} : \mathrm{R} \mapsto \mathrm{Hom}_{\mathrm{Grp}}(\mathrm{A}, \mathrm{R}^{\times}).$$

Check that $\mathrm{D(A)} \cong \mathbb{G}_{\mathrm{A}}$. *If* A *is a* finite generated *abelian group, prove that* D(A) *is isomorphic to copies of* $\mathbb{G}_m$ *and* μ_n.

Exercise 2.0.1 gives a nice, coordinate-free description of $\mathbb{G}_{\mathrm{A}}$ but we would like a characterization of "groups that look like $\mathbb{G}_{\mathrm{A}}$." To do so, we have the following simple observation:

Lemma 2.0.2. *Let* A *be a finitely generated abelian group. There are canonical bijections between:*

(1) the grouplike elements of $k[\mathrm{A}]$;
(2) characters of $\mathbb{G}_{\mathrm{A}}$;
(3) elements of A.

We have already discussed the correspondence between 1) and 2). From an element $a \in \mathrm{A}$ we get the character, described for every $\mathrm{R} \in \mathrm{CAlg}_k$ by

$$\mathrm{Hom}(\mathrm{A}, \mathrm{R}^{\times}) \xrightarrow{f \mapsto f(a)} \mathrm{R}^{\times}.$$

Write, as is standard in the literature, $\mathrm{X}^*(\mathrm{G})$ for the set of characters of the group G, i.e., the set of maps

$$\mathrm{X}^*(\mathrm{G}) := \mathrm{Hom}_{\mathrm{Grp}}(\mathrm{G}, \mathbb{G}_m);$$

This set is actually an abelian group via

$$(\chi + \chi')(g) = \chi(g) \cdot \chi'(g).$$

Lemma 2.0.2 actually produces an isomorphism of abelian groups (where $\mathrm{X}^*(\mathbb{G}_{\mathrm{A}})$ is given pointwise addition):

$$\mathrm{X}^*(\mathbb{G}_{\mathrm{A}}) \cong \mathrm{A}.$$

A definition for groups that "look like $\mathbb{G}_{\mathrm{A}}$" is:

Definition 2.0.3. A group G is **diagonalizable** if the grouplike elements of $\mathcal{O}(\mathrm{G})$ spans it as a k-vector space.

Let $\mathrm{DiagGrp} \subset \mathrm{AffGrp}_k$ be the category diagonalizable groups. Then

Theorem 2.0.4. *The functor*

$$\begin{aligned} (\mathrm{Ab}^{\mathrm{g}})^{\mathrm{op}} &\to \mathrm{DiagGrp} \qquad && \mathrm{A} \mapsto \mathrm{D(A)}; \\ \mathrm{DiagGrp} &\to (\mathrm{Ab}^{\mathrm{g}})^{\mathrm{op}} \qquad && \mathrm{G} \mapsto \mathrm{X}^*(\mathrm{G}) \end{aligned}$$

is an inverse, exact equivalence of abelian categories. In particular, all diagonalizable groups are commutative.

Remark 2.0.5. If H is a subgroup of G, then $\mathcal{O}(\mathrm{G}) \to \mathcal{O}(\mathrm{H})$ surjective so that $\mathcal{O}(\mathrm{H})$ is also spanned by grouplike elements. If $\mathrm{G} = \mathrm{D(M)}$ and $\mathrm{G} \to \mathrm{Q}$ is a quotient then its kernel takes the form $\mathrm{D}(\mathrm{M}'')$ where $\mathrm{M} \to \mathrm{M}''$ is surjective. Hence Q must be $\mathrm{D}(\mathrm{M}')$. It is not quite true that extensions of diagonalizable groups are multiplicative as there is no reason why it should be abelian. We will see that this is the only obstruction.

Hence, the study of diagonalizable groups should be thought of as "easy" and its representation theory is governed by diagonalizable representations:

Definition 2.0.6. A representation of a group G is **diagonalizable** if it can be written as a direct sum of 1-dimensional representations.

For what follows, given a character $\chi \in \mathrm{X}^*(\mathrm{G})$, write $a(\chi) \in \mathcal{O}(\mathrm{G})$ for the grouplike element that it corresponds to and write, for a representation $\rho : \mathrm{G} \to \mathrm{GL(V)}$:

$$\mathrm{V}_{\chi} := \{v \in \mathrm{V} : \rho(v) = v \otimes a(\chi).$$

Then V_{χ} is a G-stable subspace and is a 1-dimensional representation of G.

Lemma 2.0.7. *The following are equivalent:*

1. G *is diagonalizable;*
2. *every finite dimensional representation of* G *is diagonalizable, i.e., it is a direct sum of 1-dimensional representation of* G*;*
3. *every representation can be written as a direct sum of* G*-stable subspaces*

$$\mathrm{V} \cong \oplus_{\chi} \mathrm{V}_{\chi}.$$

We now describe groups of multiplicative type which are "geometrically diagonalizable."

Definition 2.0.8. A group G is **of multiplicative type** if there is an extension k'/k such that $\mathrm{G}_{k'}$ is diagonalizable.

Remark 2.0.9. Recall that we have seen non-split tori; groups of multiplicative type are then generalizations of tori.

We next characterize groups of multiplicative type. The relevant Hopf algebra notion is being **coétale**: if C is a coalgebra, then $\mathrm{C}^{\vee}$ is automatically an algebra. If C is commutative and étale then we say that C is coétale. Here's a simple definition of étale algebras: if A is a k-algebra then it is étale if for any separable closure k^s of k, we have that $\mathrm{A} \otimes_k k^s \cong k^s \times \cdots \times k^s$ where there are only finitely many members of the product.

Theorem 2.0.10. *For an algebraic group* G*, the following are equivalent:*

1. G *is of multiplicative type;*
2. G *is commutative and* $\mathrm{Hom}(\mathrm{G}, \mathbb{G}_a) = 0$*;*
3. G *is commutative and* $\mathcal{O}(\mathrm{G})$ *is coétale;*
4. G *is diagonalizable over a separable closure of* k*.*

Proof sketch. This is [**?**, Theorem 12.18]. Let us highlight some interesting features of the proof:

1. if G admits an interesting map to $\mathbb{G}_a$ $f : \mathrm{G} \to \mathbb{G}_a$, then the representation
$$a \mapsto \begin{bmatrix} 1 & f(a) \\ 0 & 1 \end{bmatrix}$$
fails to be diagonalizable over any extension. So the behavior of groups of multiplicative type is "orthogonal" to unipotent groups;
2. point (4) in particular says that diagonalizability is something that can be checked on a *separable* extension of k. Here, the point is that the splitting condition to make sure that A is étale over k can only be checked after a separable extension and not inseparable ones. More precisely if a group of multiplicative type splits over a purely inseparable extension of k, then it must already have split over k [**?**, Corollary 12.20].

□

Remark 2.0.11. An interesting feature of the theory of multiplicative groups is that to check multiplicativity one only needs to go to separable extensions.

2.1. Representation theory of groups of multiplicative type. If G is diagonalizable, then clearly its category of representations is semisimple and is completely determined by its characters: they form a complete set of simple objects. In fact a previous exercise says that the category of representation is equivalent to the category of A-graded vector spaces and the vector space of homogeneous grading a. The following result says that the same is true for groups of multiplicative type, up to Galois descent.

Theorem 2.1.1. *Let* G *be a group of multiplicative type. Then:*

1. *the category* $\mathrm{Rep}(\mathrm{G})$ *is semisimple;*
2. *let* Γ *be the Galois group of a separable closure* k^s *of* k*, then* Γ *acts continuously and naturally on* $\mathrm{X}^*(\mathrm{G}_{k^s})$*;*

(3) simple objects of $\mathrm{Rep}(\mathrm{G})$ *is in natural bijection with* Γ*-orbits of* $\mathrm{X}^*(\mathrm{G}_{k^s})$*.*

Proof. The group Γ acts on $\mathrm{X}^*(\mathrm{G}_{k^s})$ because every map $\mathrm{G}_{k^s} \to (\mathbb{G}_m)_{k^s}$ is defined over a finite separable extension of k. Actually we encourage the reader not to worry too much about continuity for now because we will always make arguments over finite separable extension; in any case the action is given by the tensoring with the map $\sigma : \mathrm{K} \to \mathrm{K}$.

Now, there exists a finite separable extension K/k with Galois group Γ (by Theorem **??**) on which the base change G_K is diagonalizable. Let V be a G-representation, then $\mathrm{V} \otimes_\mathrm{K}$ is a representation of G_K with a semilinear Γ-action. Now $\mathrm{V} \otimes \mathrm{K}$ decomposes into G_K-representations of dimension one:

$$\mathrm{K} \otimes \mathrm{V} \cong \bigoplus_{\chi \in \mathrm{X}^*(\mathrm{G}_\mathrm{K})} \mathrm{V}_\chi .$$

An element σ of Γ acts on $\mathrm{K} \otimes \mathrm{V}$ while σ moves V_χ to $\mathrm{V}_{\sigma\chi}$ where $\sigma\chi$ is the action of the first paragraph. Hence all characters which indexes a nonzero vector space above forms a single Γ-orbit. By the equivalence of Proposition **??**, this is the same datum as G-representation. Hence we see that Γ-orbits form simple objects of Rep(G).

□

2.2. Multiplicative type groups versus unipotence. We have

Proposition 2.2.1. *An algebraic group that is both unipotent and of multiplicative type is trivial.*

Proof. If G is of multiplicative type, then V must be a direct sum of simples by Theorem **??**. But then, for any unipotent group, the action on these simples must be trivial. If we apply this to a faithful representation of G (which we can), we conclude that G must be trivial. □

Corollary 2.2.2. *Let* G *be a group, if* U *is unipotent and* H *is of multiplicative type and are subgroups of* G*, then* $\mathrm{U} \cap \mathrm{H} = \{e\}$*.*

Corollary 2.2.3. *There are no nontrivial maps:*

(1) from a unipotent group to a one of multiplicative type;

(2) from a group of multiplicative type to a unipotent group.

2.3. Solvable groups. Even though there are no maps between a group of multiplicative type and a unipotent group (and vice versa), there is an interesting class of groups which are "extensions of a multiplicative group by a unipotent group." We will also quotient out this class of groups in order to get a sharper classification result.

Definition 2.3.1. Let G be an algebraic group, then:

1. a **subnormal series** of G is a finite sequence of algebraic subgroups:
$$* \lhd \mathrm{G}_s \lhd \mathrm{G}_{s-1} \lhd \cdots \mathrm{G}_0 = \mathrm{G};$$
2. a subnormal series is **normal** if G_i is normal in G.
3. G is **solvable** if there is a subnormal series such that
$$\mathrm{G}_i/\mathrm{G}_{i+1}$$
is commutative. In other words, G is solvable if it can be reconstructed via successive extensions of commutative groups.

Example 2.3.2. We write the subgroup of upper triangular matrices of GL_n as

$$\mathrm{B}_n = \begin{bmatrix} * & * & & \dots & * \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & *. \end{bmatrix}$$

These groups are always solvable; for $n = 2$:

$$e \subset \{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\} \subset \{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = \mathrm{B}_2\}$$

The subquotients are $\mathbb{G}_m \times \mathbb{G}_m$ and $\mathbb{G}_a$.

Remark 2.3.3 (Solvability and derived subgroups). In usual group theory, the derived group of a group G is exactly its commutator; we will continue this terminology. The **derived subgroup** of a group G is the intersection of all normal subgroups N such that G/N is commutative; it will be denoted by DG. One can show that G/DG is commutative. In fact, DG has an explicit model as the smallest subgroup of G containing the image of the map

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G} \qquad (g, h) \mapsto ghg^{-1}h^{-1}.$$

Set $\mathrm{D}^n\mathrm{G} := \mathrm{D}(\mathrm{D}^{n-1}\mathrm{G})$. We can then consider the derived series of G

$$\cdots \mathrm{D}^2 \lhd \mathrm{DG} \lhd \mathrm{G}.$$

This is a normal series such that the quotients are commutative, by design. One can prove that a group G is solvable if and only if the derived series terminates at a finite stage [**?**, Proposition 6.30].

By taking subs and quotients for filtrations, we can prove:

Lemma 2.3.4. *Being solvable is a property closed under taking subgroups, quotients and extensions.*

Because of this, we have the **radical** of G

$$\mathrm{R}(\mathrm{G}) \lhd \mathrm{G}$$

which is the largest normal solvable subgroup contained inside G. We will not have to consider an arbitrary solvable groups; rather we will look only at smooth, connected ones. Of the main results of Lie theory is the following:

Theorem 2.3.5 (Lie-Kolchin). *Let* G *be a smooth, connected solvable group over an algebraically close field* k. *Then there exists an embedding* $\mathrm{G} \hookrightarrow \mathrm{B}_n$.

The class of group which are subgroups of B_n are usually called **trigonalizable**:

Definition 2.3.6. A group G is called **trigonalizable** if every simple representation is of dimension one. A representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ is said to be **trigonalizable** if there exists a basis for V for which ρ acts via upper triangular matrices.

Example 2.3.7. We have already implicitly used this idea many times, but us recall the language of flags: a **flag** in a finite dimensional vector space V is a sequence of *distinct* subspaces

$$0 = \mathrm{V}_0 \subset \mathrm{V}_1 \subset \mathrm{V}_r;$$

we say that it is **maximal** if $r = \dim(\mathrm{V})$ and $\dim(\mathrm{V}_i) = i$ for all $0 \leqslant i \leqslant r$. A representation of G is trigonalizable if and only if there is a maximal flag of V which is fixed by G; note that G is then unipotent if the action on each subquotient of the flag is trivial (hence the diagonal is 1 as opposed to arbitrary invertible element).

Example 2.3.8. Unipotents and diagonalizable groups are trigonalizable. It is not quite obvious yet why looking at trigonalizable groups is a good idea except that it unifies these "opposing" notions.

Here's the standard equivalence properties of trigonalizable groups:

Proposition 2.3.9. *The following are equivalent for* G*:*

1. G *is trigonalizable;*
2. *every nonzero representation of* G *contains a one-dimensional subrepresentation.*
3. *every finite dimensional representation of* G *is trigonalizable;*

(4) G *is a subgroup of* B_N *for some* N*;*
(5) there exists a normal unipotent subgroup U *of* G *such that* G/U *is diagonalizable.*

Proof sketch. Inside B_n, the subgroup of matrices with 1's on the diagonal is a unipotent with quotient being a bunch of $\mathbb{G}_m$'s. This proves that 3 implies 4. On the other hand, if V is a simple representation of G, then we see that V^U must be nonzero; by normality again G acts on V^U via G/U and so V^U is a sum of one-dimensional representation. In particular, it contains a one-dimensional subrepresentation. □

Here's a result that characterizes groups which are trigonalizable over some extension of k as an extension of a group of multiplicative type by a unipotent:

Proposition 2.3.10. *The following are equivalent:*

(1) G *is trigonalizable over some extension of* k*;*
(2) G *contains a normal unipotent subgroup* G *such that* G/G_u *is of multiplicative type.*

If these conditions hold, then G_u *is unique and contains all other unipotent subgroups of* G*.*

Proof sketch. For the last part: if V is a unipotent subgroup, then $V \to G \to G/U$ must be trivial by Corollary **??**

□

After Theorem **??**, we see that any smooth, connected solvable group is trigonalizable over some extension of k and hence Propostion **??** furnishes a classification result; it can be sharpened as follows:

Theorem 2.3.11. *Let* k *be an algebraically close field and* G *a smooth, connected solvable group. Then*

$$G \cong G_u \ltimes T$$

where:

(1) G_u *is a normal, unipotent subgroup* G *which contains all other unipotent subgroup of* G*;*
(2) T *is a maximal torus of* G*.*
(3) the map $T \hookrightarrow G \to G/G_u$ *is an isomorphism (which implies the semidirect product decomposition).*

3. Reductive groups

We can now formulate the idea of a reductive group.

Definition 3.0.1. Let G be a connected and smooth over k. Then we say that G is **reductive** if $R_u(G_{\overline{k}}) = *$ for $\overline{k}/k$ an algebraic closure of k; if we call $R_u(G_{\overline{k}})$ the **geometric unipotent radical** then a reductive group is one whose geometric unipotent radical is trivial.

Remark 3.0.2. There is a notion of **pseudo-reductivity** which is a little bit trickier: we also ask that G is connected and smooth but that the unipotent radical $R_u(G)$ is trivial (as opposed to the geometric unipotent radical). We will not really explore this notion.

Example 3.0.3 (Multiplicative type groups). By Corollary **??**, any group of multiplicative type must have trivial geometric unipotent radical hence they are reductive. These are the simplest type of reductive groups.

Let us now explore examples. The following result is the main tool by which we can detect reductive groups:

Proposition 3.0.4. *Assume that* G *is connected, smooth and admits a faithful semisimple representation that remains semisimple after passing to the algebraic closure. Then* G *is reductive.*

PROOF. Assume that U is unipotent and is a *normal* subgroup of G. Then for any G-representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$, we have that V^{U} is G-stable. Since U is unipotent, $\mathrm{V}^{\mathrm{U}} \neq 0$. So if V is simple we must conclude that $\mathrm{V}^{\mathrm{U}} = \mathrm{V}$. Picking any semisimple representation, which is a sum of simples, which is also faithful gives us that U is trivial.

□

As we see from the proof above, reductive groups can have unipotent subgroups as long as they are not normal.

EXAMPLE 3.0.5. The standard representation of GL(V), i.e., GL(V) acting on itself is a semisimple, faithful representation. Hence GL_n is reductive for $n \geqslant 0$.

EXERCISE 3.0.6. *Prove, via a similar method or otherwise, that the* SO, SL *and* Sp *are all reductive.*

EXERCISE 3.0.7. *Let* $n, m \geqslant 0$. *Consider the subgroup* G *of* GL_{n+m} *given by*

$$\{\begin{bmatrix} \mathrm{A}_m & * \\ 0 & \mathrm{B}_n \end{bmatrix}\}.$$

Compute $\mathrm{G}/\mathrm{R}_u(\mathrm{G})$.

3.1. Borel subgroups. We now begin to explain the structure theory of reductive groups. The "wings of the butterfly" are called **Borel subgroups**. From now on, we let k be an *algebraically closed field* and assume that G be a smooth affine algebraic group. Among all subgroups of G we define the **condition R** (where R stands for radical) on a subgroup H as the condition that H is smooth, connected solvable subgroup.

DEFINITION 3.1.1. A **Borel subgroup** is a maximal subgroup of G among those satisfying condition R.

EXAMPLE 3.1.2. Consider a Borel subgroup B of GL(V). By Theorem **??**, there is a basis for V under which $\mathrm{B} \subset \mathrm{B}_n$ and $\mathrm{B} = \mathrm{B}_n$ by maximality. There is a better way to say this: Borel subgroups are exactly the stabilizers of maximal flags in V.

EXERCISE 3.1.3. *Compute the Borel subgroups of the* SO_n*'s.*

Here is the main theorem about Borel's

THEOREM 3.1.4. *A group* B *is maximal among (not necessarily normal) subgroups satisfying* R *if and only if* G/B *is proper.*

This result is a conjunction of the following lemma

LEMMA 3.1.5. *If* H *is a subgroup of dimension maximal among all subgroups satisfying condition* R*, then* G/H *must be proper. Furthermore, all Borel subgroups of* G *are conjugate by an action of* $\mathrm{G}(k)$.

and the eponymous

THEOREM 3.1.6 (Borel's fixed point theorem). *Let* k *be algebraically closed and* G *a solvable group acting on a proper* k*-scheme* X. *Then* $\mathrm{X}(k)^{\mathrm{G}} \neq \emptyset$.

PROOF. Let us induct on the dimension of G. There is nothing to prove when G is dimension zero (it is the trivial group so the assertion is that $\mathrm{X}(k)$ is nonempty, which is true by the Nullstellesatz). Now, if $\dim \mathrm{G} = 1$, then G must either be $\mathbb{G}_a$ or $\mathbb{G}_m$. Fix a point $x_0 \in \mathrm{X}(k)$ and consider the translation

$$\mathrm{G} \to \mathrm{X} \qquad g \mapsto gx_0;$$

we claim that this map extends to a G-equivariant map $\mathbb{P}^1 \to \mathrm{X}$ where G acts on $\mathrm{G} \subset \mathbb{P}^1$ by translation. Indeed, we have a compactification $\mathrm{G} \hookrightarrow \mathbb{P}^1$ in such a way that the action of G on itself by translation extends to the new points by trivial action. The valuative criterion then extends the map in an equivariant way (this is to be proved below). Therefore the image of ∞ is actually a k-rational point which is not moved by G.

Now, assume that $d > 1$. There then exists $\mathrm{H} \lhd \mathrm{G}$ which is normal, and H is solvable. By induction we can find a point $x' \in \mathrm{X}(k)$ which is H-fixed. The orbit map at x', $\mathrm{G} \to \mathrm{X}$ then induces an G-equivariant map $\mathrm{G}/\mathrm{G}' \to \mathrm{X}$ by universal properties of quotients. But the quotient is $\mathbb{G}_a$ or $\mathbb{G}_m$ so we are reduced to the case of the previous paragraph. ☐

This also gives us a proof of Lie-Kolchin

Proof of Theorem **??**. Let F be a maximal flag, let us denote the functor

$$\mathrm{Flag}_{\mathrm{V}} : \mathrm{R} \mapsto \{0 \subset \mathrm{F}_0 \subset \mathrm{F}_r = \mathrm{V} \otimes \mathrm{R} : \mathrm{F}_i \text{ is a direct summand of } \mathrm{R} \otimes \mathrm{V}_i\}.$$

Here's a key geometric result: the functor Flag is represented by a smooth projective (whence proper) variety.

Let V be a faithful representation of G; there is an induced action of G on $\mathrm{Flag}_{\mathrm{V}}$. By Borel's fixed point theorem, there exists a fixed point of $\mathrm{Flag}_{\mathrm{V}}(k)$ which is stabilized by G. Therefore, we conclude that the representation is trigonalizable and hence G is since the representation is faithful.

☐

Assuming the key lemma we given

Proof of Theorem **??**. Before the start of the proof, we do not know that maximal subgroups have the same dimension. But indeed, if B is maximal, then it must of maximal dimension. Therefore if B is maximal satisfying R, then it must be of maximal dimension too, hence Lemma **??** says that G/H must be proper. Conversely, assume that G/B is proper, we want to prove that B is maximal. Let H be a subgroup satisfying R, then we claim that

$$\mathrm{H} \subset g\mathrm{B}g^{-1}$$

for some $g \in \mathrm{G}(k)$. Indeed, H acts on G/B and the latter is proper hence H fixed a coset in $\mathrm{G}/\mathrm{B}(k)$. Such a coset is of the form $g\mathrm{B}$ and the fact that H fixes $g\mathrm{B}$ means that $\mathrm{H} \subset g\mathrm{B}g^{-1}$. Now choose H to be a maximal subgroup satisfying R and hence $\mathrm{H} = g\mathrm{B}g^{-1}$, but this means that B and H have the same dimension, hence it is maximal with respect to dimension among subgroup satisfying R and hence B is maximal.

Let us prove the second statement. Say B is already known to be a Borel. If H is a subgroup satisfying R, then

☐

This encourages us to look for other groups having the property that its quotient is proper:

Definition 3.1.7. A **parabolic** subgroup P is one such that G/P is proper.

3.2. Maximal tori.

In Grothendieck's picture of the butterfly, a torus is the body of the butterfly. We will now prove that this body exists nontrivially whenever G is not unipotent; we again remain the algebraically closed setting.

Proposition 3.2.1. *If* G *is a smooth connected group and is* not *unipotent, then there is a nontrivial torus* $\mathbb{G}_m \subset \mathrm{G}$.

Theorem 3.2.2. *Any two maximal tori are conjugate via an element of* $\mathrm{G}(k)$.

4. Semisimple groups

Definition 4.0.1. If k is any field, the **rank** of G is the rank of the maximal torus in $\mathrm{G}_{\overline{k}}$. The **semisimple rank** of G is the rank of $\mathrm{G}_{\overline{k}}/\mathrm{R}(\mathrm{G}_{\overline{k}})$.

CHAPTER 5

Towards classification theorems

1. Root datum and root systems

We explain the combinatorial structure that appears in the theory of reductive groups. Let X be a finitely generated free $\mathbb{Z}$-module; we than have its dual $X^\vee := \mathrm{Hom}_{\mathrm{AbGrp}}(X, \mathbb{Z})$ which comes with a nondegenerate bilinear pairing

$$\langle , \rangle : X \times X^\vee \to \mathbb{Z}(x, f) \mapsto f(x).$$

Definition 1.0.1. A **root datum** is a triple

$$\mathcal{R} = (X, R, \alpha \mapsto \alpha^\vee)$$

where X is a finitely generated free ablelian group, $R \subset X$ is a finite set and

$$R \to X^\vee \qquad \alpha \mapsto \alpha^\vee$$

is a map satisfying:

(RD1) under the pairing explained above, we have
$$\langle \alpha, \alpha^\vee \rangle = 2;$$

(RD2) Define, for each $\alpha \in R$,
$$s_\alpha : X \to X$$
given by
$$x \mapsto x - \langle x, \alpha^\vee \rangle \alpha.$$
Then $s_\alpha(R) \subset R$;

(RD3) the subgroup of $\mathrm{Aut}_{\mathbb{Z}}(X)$ generated by s_α is finite; the latter is called the **Weyl group** of $\mathcal{R}$ written as $W(\mathcal{R})$.

Any element $\alpha \in R$ is called a **root** and $\alpha^\vee$ is a coroot. We say that root datum is **reduced** if for each $\alpha \in R$, the only multiples of α in R are $\pm\alpha$.

Remark 1.0.2 (Reflections). We motivate the definition of s_α. Let V be a finite dimensional vector space over a field k, a **reflection** is an endomorphism

$$s : V \to V$$

which fixes a hyperplane $H \subset V$ (codimension one subspace of V) and acts as -1 on the complementary line. If α is a vector such that $s(\alpha) = -\alpha$ then $V = H \oplus \langle \alpha \rangle$. In fact s is determined by H, α and the fact that $s^2 = \mathrm{id}$; here's a sharper version of this statement:

Lemma 1.0.3. *If $\alpha^\vee$ is an element of $V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$, then the map*

$$s_\alpha : x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$$

is a reflection with $s_\alpha(\alpha) = -\alpha$ and every reflection is of this form with $\alpha^\vee$ uniquely determined.

We have used the geometrically evocative words like "reflection." This leads us to the notion of a **root system** which should be thought of as a visual realization of sort of a root datum:

Definition 1.0.4. A **root system** R is a vector space over a field k of characteristic zero (most likely, in our context, this is $\mathbb{R}$) such that:

(1) R is finite, spans V and does not contain zero;

(2) for each $\alpha \in \mathrm{R}$ there is a reflection s_α with vector α such that $s_\alpha(\mathrm{R}) \subset \mathrm{R}$;
(3) for all $\alpha, \beta \in \mathrm{R}$ the element

$$s_\alpha(\beta) - \beta \in \mathbb{Z}\alpha.$$

The **Weyl group**, $\mathrm{W(R)}$, of a root system is the group of k-linear automorphisms of V generated by the reflections.

We can go from a root datum to a root system: take $\mathrm{X}_{\mathbb{Q}}$ which is not a $\mathbb{Q}$-vector space and set V to be the $\mathbb{Q}$-subspace of V spanned by R. Then (V, R) is a root system with the same corresponding names of α and $\alpha^\vee$. There are other adjectives to ensure a correspondence between root system and root datum which we will discuss later but let us look at some examples.

1.1. A root datum out of reductive groups. Let G be a reductive group over k and fix a maximal torus T. The pair (G, T) is said to be a **split reductive group** if T is a split torus. Consider the Lie algebra which comes graded by the action of the torus:

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in \mathrm{X}^*(\mathrm{T})} \mathfrak{g}_\alpha;$$

for most of the α's, $\mathfrak{g}_\alpha = 0$. The subset of $\mathrm{X}^*(\mathrm{T})$ for which this is not the case is written as

$$\Phi(\mathrm{G}, \mathrm{T}) \subset \mathrm{X}^*(\mathrm{T})$$

and we say that such an α is a **root**.

Recall that we have seen the Weyl group defined as $\mathrm{W}(\mathrm{G}, \mathrm{T}) := \mathrm{N_G}(\mathrm{T})/\mathrm{C_G}(\mathrm{T})$.

LEMMA 1.1.1. *Let* G *be a reductive group*

(1) a torus T *is maximal if and only if it is central;*
(2) if T *is maximal and split, then* $\mathrm{W}(\mathrm{G}, \mathrm{T})$ *is a finite constant group.*

PROOF SKETCH. The first claim is a consequence of a Chevalley's formula for the unipotent radical which is not obvious [**?**, Theorem 17.56], which we will discuss later. For the second fact: $\mathrm{W}(\mathrm{G}, \mathrm{T})$ acts faithfully on the torus and hence on $\mathrm{X}^*(\mathrm{T})$ and the latter is a constant group. □

The underlying finitely generated abelian group of the root datum that we will extract will obviously be $\mathrm{X}^*(\mathrm{T})$ and $\Phi(\mathrm{G}, \mathrm{T})$ will be the roots. Let us look at some examples first before going further:

EXAMPLE 1.1.2 ($\mathrm{GL}_2, \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$). For this split reductive group we have that

$$\mathrm{X}^*(\mathrm{T}) = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2,$$

where

$$(n\chi_1 + m\chi_2) \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \alpha_1^n \cdot \alpha_2^m.$$

We know that

$$\mathfrak{gl}_2 = \mathrm{M}_2$$

and T acts on M_2 by conjugation:

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \cdot_{\mathrm{act}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & \frac{\alpha_1}{\alpha_2} b \\ \frac{\alpha_2}{\alpha_1} c & d. \end{bmatrix}$$

Since we have a basis of M_2 given by $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4$, to compute its roots we just have to look at the action: we see that

$$\mathfrak{g}_o = k\mathrm{E}_1 \oplus k\mathrm{E}_2$$

where T acts trivially. On the other hand

$$\chi_1 - \chi_2 \cdot_{\mathrm{act}} k\mathrm{E}_2 = \tfrac{\chi_1}{\chi_2} k\mathrm{E}_2;$$

and

$$\chi_2 - \chi_1 \cdot_{\mathrm{act}} k\mathrm{E}_2 = \tfrac{\chi_2}{\chi_1} k\mathrm{E}_2.$$

How do we understand the roots $\chi_1 - \chi_2$ and $\chi_2 - \chi_1$; they are clearly the **conjugate** of one another. In fact, in a homework exercise, we have seen this symmetry is a consequence of the Weyl group action on the Lie algebra. The lemma that explains this is:

Lemma 1.1.3. *If $s \in \mathrm{W}(\mathrm{G}, \mathrm{T})$ and α is a root. Then, via the action on $\mathrm{X}^*(\mathrm{T})$ induced by the action on T, $s\alpha$ is a root as well and that $\mathfrak{g}_{s\alpha} = s\mathfrak{g}_\alpha$ via the adjoint action.*

This is [**?**, Proposition 21.1]. If we believe that $\mathrm{W}(\mathrm{G}, \mathrm{T})$ is really the Weyl group of the root system, then this verifies $s_\alpha(\mathrm{R}) \subset \mathrm{R}$ for all α. In any case, the roots of $(\mathrm{GL}_2, \mathrm{T})$ are given by $\{\pm(e_1 - e2)\} \subset \mathbb{Z}^{\oplus 2}$.

Example 1.1.4. For $\mathrm{G} = \mathrm{SL}_2$ with the diagonal $\mathbb{G}_m$, we have seen (in a previous lecture) that the roots are $\{2, -2\} \subset \mathbb{Z}$.

Example 1.1.5. Consider the group PGL_n defined via the sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

If $n = 2$ we set

$$\mathrm{T} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} / \begin{bmatrix} t & 0 \\ 0 & t. \end{bmatrix}$$

Then $\mathrm{X}^*(\mathrm{T}) = \mathbb{Z}$ and generated by the character

$$\chi(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}) = \tfrac{\alpha_1}{\alpha_2}.$$

The action on $\mathfrak{pgl}_2 = \mathrm{M}_2/\lambda(\mathrm{E}_1 + \mathrm{E}_4)$ is given by conjugation again and we see that the roots are $\{1, -1\} \subset \mathbb{Z}$.

1.2. Formulating the classification theorem. We want to formulate the classification theorem. Instead of trying to straight up prove it, we will work through a tricky example (even though it has an incarnation in the problem set): the reductive group

$$(\mathrm{GL}_3, \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3. \end{bmatrix})$$

Easy enough: the Lie algebra is given by $\mathrm{M}_3(k)$ and, just as we did in GL_2, we can figure out the roots as a subset of $\mathrm{X}^*(\mathrm{T}) = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2 \oplus \mathbb{Z}\chi_3$.

Exercise 1.2.1. *Indeed the roots are given by $\{\chi_i - \chi_j : i \neq j\}$ for a total of six roots.*

So we write down

$$\mathfrak{sl}_3 = \mathfrak{g}_0 \bigoplus_{\alpha \in \Phi(\mathrm{G}, \mathrm{T})} \mathfrak{g}_\alpha$$

and we can in fact figure out $\mathfrak{g}_\alpha$ as

$$g_{\chi_i - \chi_j} = k\mathrm{E}_{ij}$$

so for example

$$g_{\chi_1 - \chi_2} = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{bmatrix}$$

At this point it is reasonable to ask: is g_α actually the Lie algebra of a subgroup? In fact it is: for $g_{\chi_1 - \chi_2}$ this is the subgroup of SL_3 given by

$$\{\begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1, \end{bmatrix}\}$$

a unipotent! More precisely, the map

$$\mathbb{G}_a \to \mathrm{GL}_3 \qquad x \mapsto \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

induces the inclusion $\mathfrak{g}_{\chi_1 - \chi_2} \subset \mathfrak{g}$.

Lemma 1.2.2. *For each root* α*, there exists a morphism*

$$\mathbb{G}_a \to \mathrm{G}$$

such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ *for all* $t \in \mathrm{T}$*. Its image is the unipotent subgroup* U_α *for which* $\mathrm{Lie}(\mathrm{U}_\alpha) = \mathfrak{g}_\alpha$

We will go back to this unipotent later on, but let's check out the Weyl group action. This is S_3 the symmetric group on 3 letters. This identifies with the permutation matrices. We want to establish a relationship between the two groups called the Weyl groups. To proceed, we need to understand coroots: remember that a root datum does consist of a map $\alpha \mapsto \alpha^\vee : \mathrm{R} \to \mathrm{X}^\vee$; well the dual of $\mathrm{X}^*(\mathrm{T})$ are the cocharacters $\mathrm{X}_*(\mathrm{T}) = \{\lambda : \mathbb{G}_m \to \mathrm{T}\}$. In this case

$$\mathrm{X}_*(\mathrm{T}) = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \oplus \mathbb{Z}\lambda_3$$

where

$$\lambda_1(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda_2(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda_3(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix}$$

The pairing is the obvious one:

$$\langle \chi_i, \lambda_j \rangle = \delta_{ij}.$$

So what is the map

$$\alpha \mapsto \alpha^\vee?$$

Well, the pairing determines this: since we need $\langle \alpha, \alpha^\vee \rangle = 2$ we can guess what $\alpha^\vee$ is. For example:

$$(\chi_1 - \chi_2)^\vee = \lambda_1 - \lambda_2.$$

But we want to do this geometrically and also compute what s_α is. To do this, we "concentrate information" to a single root. For a root α, we set

$$\mathrm{T}_\alpha := \ker(\alpha)^\circ_{\mathrm{red}} \qquad \mathrm{G}_\alpha = \mathrm{C}_\mathrm{G}(\mathrm{T}_\alpha).$$

Lemma 1.2.3. *The pair* $(\mathrm{G}_\alpha, \mathrm{T}_\alpha)$ *is a split reductive group and* $\mathrm{Lie}(\mathrm{G}_\alpha) \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.

In our example,

$$\mathrm{T}_{\chi_i - \chi_j} = \{t \in \mathrm{T} : \alpha_i = \alpha_j\};$$

and we can describe $\mathrm{C}_\mathrm{G}(\mathrm{T}_{\chi_i - \chi_j})$ easily too; for example:

$$\mathrm{C}_\mathrm{G}(\mathrm{T}_{\chi_1 - \chi_2}) = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & *. \end{bmatrix}$$

These are isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_1$. In fact

$$\mathrm{T}_{\chi_i - \chi_j} = \mathrm{T}_{\chi_j - \chi_i} \qquad \mathrm{G}_{\chi_i - \chi_j} = \mathrm{G}_{\chi_j - \chi_i}.$$

There is something special about these groups.

Lemma 1.2.4. *The Weyl group of* $(\mathrm{G}_\alpha, \mathrm{T}_\alpha$ *is always just* $\mathbb{Z}/2$.

APPENDIX A

Some algebraic geometry

1. Schemes as Topological Spaces

The starting point for algebraic geometry is the study of shapes that are described by polynomial equations. More precisely, given some polynomial functions $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ we're interested in the "algebraic set" $\mathrm{V}(f_1, \dots, f_m) = \{x \in k^n \mid f_1(x) = \dots = f_m(x) = 0\}$. The first thing to note is that the set $\mathrm{V}(f_1, \dots, f_m)$ only depends on the ideal I that $f_1, \dots, f_m$ generate in $k[x_1, \dots, x_n]$.

Exercise 1.0.1. *Prove that if $f_1, \dots, f_m$ and $g_1, \dots, g_k$ generate the same ideal* I *in $k[x_1, \dots, x_n]$ then $\mathrm{V}(f_1, \dots f_m) = \mathrm{V}(g_1, \dots g_k)$.*

We can rephrase the above by saying that there is a map

$$\left\{\begin{array}{l}\text{Rings R with a}\\ \text{surjection from } k[x_1, \dots x_n]\end{array}\right\} \to \{\text{Algebraic sets in } k^n\}. \tag{1.0.2}$$

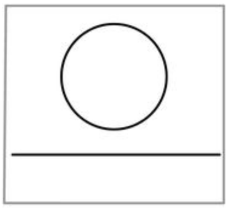
Our goal now, roughly, is to find a way of constructing an algebraic set as an abstract topological space using just the ring R. Moreover, we want this procedure to assign $\mathbf{A}^n$ to the ring $k[x_1, \dots, x_n]$, and we want the surjection $k[x_1, \dots x_n] \to \mathrm{R}$ to give the data of an embedding of the abstract topological space into $\mathbf{A}^n$.

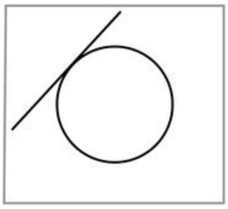
Now, though, we run into the problem that algebraic sets are not exactly the right geometric objects to consider if one wants to do algebraic geometry. Let's see why. Consider the following theorem, which could be formulated by an observant high school student.

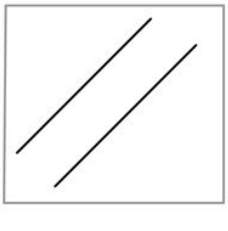
Theorem 1.0.3 (Bezout). *A degree n plane curve intersects a degree m plane curve at nm points.*

The problem is that, interpreted naively (that is, in terms of algebraic sets), this theorem is false, as the following examples show:

222 Apdx A Image 1.jpg 222 Apdx A Image 1.jpg







One wants a foundation for algebraic geometry where Bezout's theorem (among other elementary results) is true without exceptions. As the examples above show, in a theory of algebraic geometry that satisfies this constraint, phrases like "degree n plane curve" and "intersection" must refer to something different than their usual meanings.

In the first example above, the circle and the line fail to intersect at a point with real coordinates. However, viewed as equations over the complex numbers, the circle and the line *do* intersect. So we see that even if a curve is specified by equations with real coefficients, we will still need to keep track of solutions to these equations over the complex numbers.

Definition 1.0.4. The **Zariski space** $\operatorname{Spec} \mathrm{R}$ is a topological space associated to a commutative ring R. It has the set of prime ideals of R as its underlying set of points. The closed subsets of $\operatorname{Spec} \mathrm{R}$ are the sets $\mathrm{V(I)}$ as I ranges over all ideals of R, where $\mathrm{V(I)}$ is the set of prime ideals containing I.

Example 1.0.5. Suppose $\mathrm{R} = \mathbf{R}[x]$. Then the points of $\operatorname{Spec} \mathrm{R}$ fall into three categories:

- $(x-a)$ for $a \in \mathbf{R}$.
- $(x-\lambda)(x-\overline{\lambda})$ for $\lambda \in \mathbf{C} \smallsetminus \mathbf{R}$.
- (0).

With this definition in hand, we can already define some important properties of schemes (the properties that only depend on the underlying topological space).

Definition 1.0.6. A topological space X is **irreducible** if for all pairs of closed subsets $\mathrm{Z}_1, \mathrm{Z}_2 \subseteq \mathrm{X}$, if $\mathrm{Z}_1 \cup \mathrm{Z}_2 = \mathrm{X}$ then either $\mathrm{Z}_1 = \mathrm{X}$ or $\mathrm{Z}_2 = \mathrm{X}$.

Definition 1.0.7. The **Krull dimension** of a topological space X is the length d of the longest strictly increasing chain $\mathrm{Z}_0 \subset \mathrm{Z}_1 \subset \ldots \subset \mathrm{Z}_d$ of irreducible closed subspaces of X.

Exercise 1.0.8. *Show that* $\operatorname{Spec} \mathbf{R}[x]$ *is irreducible and has Krull dimension 1.*

Proposition 1.0.9. *Given a map* $f : \mathrm{A} \to \mathrm{B}$ *one obtains a continuous map* $f^{op} : \operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{A}$*, which sends a prime ideal to its preimage.*

Exercise 1.0.10. *Check that the above map is actually continuous. Show that* f^{op} *is a closed embedding if* f *is surjective.*

Theorem 1.0.11 (Hilbert's Nullstellensatz). *Let* L/K *be a field extension where* L *is finitely generated (as an algebra) over* K*. Then* L/K *is a finite field extension (*L *is a finite-dimensional* K*-vector space).*

Exercise 1.0.12. *Show that when* k *is algebraically closed the maximal ideals of* $k[x_1, \ldots, x_n]$ *are in bijection with the elements of* k^n*.*

2. Sheaves

At this point we have managed to construct $\operatorname{Spec} \mathrm{R}$ abstractly, without reference to an embedding into $\mathbf{A}^n$, and we have included enough points so that Bezout's theorem still holds even over non-algebraically closed fields. Now let's ask the following question, which is (maybe surprisingly) related to the second example above where Bezout's theorem fails: is there a way to start from an algebraic set in k^n and recover the

In the second example above where Bezout's theorem fails, the line is tangent to the circle. In this situation, even over the complex numbers there is just a single intersection point. Intuitively, though, the intersection "has multiplicity two". Let's see how to formalize this. In an appropriate coordinate system (rotated relative to how the picture was drawn), we can describe the circle as the points where $x^2 + y^2 = 1$ and we can describe the line as the points where $y = 1$. Note that the polynomial functions on the circle

Definition 2.0.1. Let X be a topological space, then the **site** associated to X, which we write $\mathcal{C}_\mathrm{X}$ is the category whose objects are the open subsets of X, so that there is an arrow (and only one arrow) from U to V if and only if $\mathrm{U} \subseteq \mathrm{V}$.

DEFINITION 2.0.2. A **presheaf** (of sets) on a topological space X is a functor $\mathcal{C}_X^{op} \to \mathrm{Set}$. A presheaf of abelian groups on X is a functor $\mathcal{C}_X^{op} \to \mathrm{Ab}$, and so on.

DEFINITION 2.0.3. A **sheaf** (of sets) on a topological space X is a presheaf $\mathcal{F}$ so that for every open cover $\{U_\alpha\}$ of a an open subset U, the diagram

$$\mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is an equalizer.

DEFINITION 2.0.4. A **ringed space** is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$. A map of ringed spaces $f : X \to Y$ is the data of a continuous map of topological spaces from X to Y (also written f) and a map of sheaves $f^*\mathcal{O}_Y \to \mathcal{O}_X$.

NOTATION 2.0.5. Given a ring R and an element $f \in R$, we write $D(f)$ for the open subspace $\operatorname{Spec} R \smallsetminus V(f)$.

EXERCISE 2.0.6. *Show that as f ranges over the elements of* R *the subsets* $D(f)$ *form a base for the Zariski topology on* Spec R*.*

PROPOSITION 2.0.7. *There is a unique sheaf of rings* $\mathcal{O}$ *on* Spec R *so that for every* $f \in R$*,*

$$\mathcal{O}(D(f)) = R_f.$$

DEFINITION 2.0.8. The **affine scheme** Spec R is the ringed space $(\operatorname{Spec} R, \mathcal{O})$.

DEFINITION 2.0.9. Let X be a topological space, and let $p \in X$ be a point in X. If $\mathcal{F}$ is a presheaf over X then the **stalk** of $\mathcal{F}$ at p, written $\mathcal{F}_p$, is $\operatorname{colim}_U \mathcal{F}(U)$, where the colimit ranges over all open sets U containing p.

DEFINITION 2.0.10. A **locally ringed space** is a ringed space so that the stalk at every point is a local ring. A map $f : X \to Y$ of locally ringed spaces is a map of ringed spaces so that for every $x \in X$ the induced map on stalks $(f^*\mathcal{O}_Y)_x \to (\mathcal{O}_X)_x$ sends the maximal ideal of $(\mathcal{O}_Y)_{f(x)}$ to the maximal ideal of $(\mathcal{O}_X)_x$.

PROPOSITION 2.0.11. *There is a natural bijection between maps of locally ringed spaces from* $\operatorname{Spec} A \to \operatorname{Spec} B$ *and maps of rings* $B \to A$*.*

DEFINITION 2.0.12. A **scheme** X is a ringed space which has an open cover by affine schemes. That is to say...

3. Properties of Schemes

DEFINITION 3.0.1. A scheme is **reduced** if it has an open affine cover by $\operatorname{Spec} A_i$, where A_i is reduced.

DEFINITION 3.0.2. A scheme is **integral** if it is irreducible and reduced.

DEFINITION 3.0.3. A morphism $f : X \to Y$ of schemes is **locally of finite type** if Y has an open affine cover by $\operatorname{Spec} A_i$ and and each preimage $f^{-1}(\operatorname{Spec} A_i)$ has an open affine cover by $\operatorname{Spec} B_{ij}$ such that B_{ij} is a finitely-generated A_i-algebra for all i, j.

3.1. Separation axioms.

Recall that:

DEFINITION 3.1.1. A morphism of schemes $f : X \to Y$ is said to be **separated** if the diagonal map $\Delta_f : X \to X \times_Y X$ is a closed immersion. We say that a scheme X is **separated** if the morphism $X \to \operatorname{Spec} \mathbf{Z}$ is.

EXAMPLE 3.1.2. Any affine scheme $X = \operatorname{Spec} A$ is separated: the diagonal corresponds to the ideal $\ker(m) \subset A \otimes_k A$ where m is the multiplication map. On the other hand the bug-eyed point is the ultimate example of a non-separated scheme.

3.2. Regularity properties.

Definition 3.2.1. A Noetherian local ring $(\mathrm{A}, \mathfrak{m})$ is **regular** if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathrm{A}$, where $\dim \mathfrak{m}/\mathfrak{m}^2$ is the dimension as an $\mathrm{A}/\mathfrak{m}$ vector space and $\dim \mathrm{A}$ is the Krull dimension.

Remark 3.2.2. Let $\kappa = \mathrm{A}/\mathfrak{m}$. We always have an inequality

$$\dim \mathrm{A} \leqslant \dim \mathfrak{m}/\mathfrak{m}^2,$$

and so a regular local ring is one in which this becomes an equality. To get a taster of the phenomenon of regularity we give discuss a couple of examples:

1. Consider the ring $\mathbb{C}[x]/(x^2)$; this is a thickened point and has dimension 0. On the other hand $\mathfrak{m}/\mathfrak{m}^2 = \mathbb{C}\{x\}$ and so has dimension one.
2. Consider the cuspidal curve $\mathrm{X} = \operatorname{Spec} \mathbb{C}[x, y]/(x^2 = y^3)$. We look at the local ring at the point corresponding to (x, y). Then the dimension is of course one but

$$\mathfrak{m}/\mathfrak{m}^2 = \tfrac{(x,y)}{(x^2,xy,y^2)} = \mathbb{C}\{x, y\}$$

which is two dimensional. At another point, say $(x-1, y-1)$, the computation becomes different; indeed:

$$x^2 - 1 = y^3 - 1$$

and so

$$(x-1)(x+1) = (y-1)(y^2 + y + 1).$$

But note that in $\mathbb{C}[x, y]/(x^2 = y^3)_{(x-1,y-1)}$ the polynomials $x + 1$ and $y^2 + y + 1$ are invertible, therefore

$$\mathfrak{m} = (x-1, y-1) = (y-1);$$

in this case

$$\mathfrak{m}/\mathfrak{m}^2 = \tfrac{(y-1)}{((y-1)^2,(x-1)(y-1),(x-1)^2)}$$

and thus is 1-dimensional.

Definition 3.2.3. A scheme is **regular** if every local ring $\mathcal{O}_{\mathrm{X},x}$ is regular.

Definition 3.2.4. A scheme X over a field k is **smooth** if it is locally of finite type over k and geometrically regular, that is to say: if $\overline{k}$ is any separable algebraic closure of k, then $\mathrm{X}_{\overline{k}}$ is a regular scheme.

We offer three further ways to think about smoothness (in fact, there are many ways to think about smoothness!):

Remark 3.2.5 (Cohen structure theorem). The notion of smoothness captures the idea from differential topology that "locally everything looks like affine space." However, due to the fact that Zariski open sets are big, it is not literally the case that regular local rings are isomorphic to local rings of affine space. We have to *zoom in further.* To make this precise, let us recall the idea of complete local rings: we say that $(\mathrm{A}, \mathfrak{m})$ is a **complete local ring** if the map

$$\mathrm{A} \to \lim \mathrm{A}/\mathfrak{m}^n$$

is an isomorphism; the target ring is an example of *completion.* Let k be a field suppose that A contains k, then an example of A which is complete is the local ring of power series in n-variables $(k[[t_1, \cdots, t_n]], (t_1, \cdots, t_n))$. Completing closed points of affine spaces gives these local rings. The Cohen structure theorem is then the fact that any regular local ring completes to one of this form (a reference is [**?**, Tag 0C0S]):

Theorem 3.2.6. *Let* $(\mathrm{A}, \mathfrak{m})$ *be a complete, noetherian local ring and assume that there exists a field* k *and a map* $k \to \mathrm{A}$ *such that the induced map* $k \to \mathrm{A}/\mathfrak{m}$ *is an isomorphism, then the following are equivalent:*

1. A *is regular;*
2. $\mathrm{A} \cong k[[t_1, \cdots t_n]]$.

Therefore, the intuition for smooth k-schemes is the following: we may first let k be algebraically closed (after checking appropriate finite type hypotheses of course), then we may just check that local rings of closed points (it suffices to check smoothness on closed points — can you explain why?) are isomorphic to power series rings.

Remark 3.2.7 (Jacobian criterion). Sometimes we might be presented with explicit equations describing a scheme, at least locally. So let us concentrate on the affine setting: if $k \to \mathrm{A}$ is a morphism, then if A is finite type we can write

$$\mathrm{A} = k[x_1, \cdots, x_n]/(f_1, \cdots, f_c).$$

From this, we may form the matrix of partials called the **Jacobian**

$$\left(\frac{\partial f_i}{\partial x_j}\right).$$

We say that $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ is smooth at a point $x \in \mathrm{X}$ if the jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)$ has full rank. One of the nice things about the Jacobian criterion is that it makes it obvious that smoothness is an *open condition*: if X is smooth at a closed point $x \in \mathrm{X}$ then it is smooth at any point $y \in \mathrm{X}$ such that $x \in \overline{\{y\}}$. Indeed, the condition that the Jacobian is full rank is equivalent to the non-vanishing of minors, which is an open condition. This leads to the following result (which we will use in the main text):

Lemma 3.2.8. *Let k be a perfect field and* X *be a non-empty finite type k-scheme which is reduced. Then there is a dense open subscheme of* X *which is smooth.*

Proof. We may assume that X is irreducible. We note that if X is integral then X is locally modeled by $k[\mathrm{T}_1, \cdots, \mathrm{T}_d, \mathrm{T}]/(g)$ where g is a separable irreducible polynomial $g \in k(\mathrm{T}_1, \cdots, \mathrm{T}_d)[\mathrm{T}]$ with coefficients in $k[\mathrm{T}_1, \cdots, \mathrm{T}_d]$; here we use perfectness[1]. Indeed, let K be the function field of X, then since k is a perfect field then K must be a finitely generated separable extension of k and so we can find a transcendence basis $\mathrm{T}_1, \cdots, \mathrm{T}_d$ over k such that the extension $k(\mathrm{T}_1, \cdots, \mathrm{T}_d) \subset \mathrm{K}$ is finite and separable. The primitive element theorem then says that K is generated over $k(\mathrm{T}_1, \cdots, \mathrm{T}_d)$ by a single element α; we may assume that the minimal polynomial of α is in $k[\mathrm{T}_1, \cdots \mathrm{T}_d] \subset k(\mathrm{T}_1, \cdots \mathrm{T}_d)$ after clearing denominators. In any event, there is a dense open of X given by $\operatorname{Spec} k[\mathrm{T}_1, \cdots, \mathrm{T}_d, \mathrm{T}]/(g)$ where g is a separable irreducible polynomial with coefficients in $k(\mathrm{T}_1, \cdots \mathrm{T}_d)$.

The claim now becomes verifying the Jacobian criterion: if $g_i = \partial g/\partial \mathrm{T}_i \in k[\mathrm{T}_1, \cdots, \mathrm{T}_{d+1}]$, then g does not divided g_i. By degree reasons, the only way this can *not* happen is if all the g_i's vanish. This can't happen in characteristic zero. In characteristic $p > 0$ the vanishing of all the partial derivatives mean that $g = f^{p^n}$. But this means that K is not separable over $k(\mathrm{T}_1, \cdots, \mathrm{T}_d)$ which is false by assumption

□

The last perspective we take on smoothness is the Grothendieck criterion:

Remark 3.2.9 (Grothendieck criterion). A **first order thickening** is a surjective morphism $\mathrm{A} \to \mathrm{B}$ of rings such that the ideal kernel I satisfy $\mathrm{I}^2 = 0$. The geometric picture

---

[1] Some comments on field theory. Recall that a field is said to be **perfect** if every field extension is separable over k [**?**, Tag 05DU]. Since the notion of separability might be confusing in the nonalgebraic setting we recall that a *finitely generated* extension (there's also a notion for non finitely generated extension but we ignore it for now) K/k is said to be separable *over* k if we can find a transcedence basis $\{x_1, \cdots, x_n\}$ such that K is separable algebraic over $k(x_1, \cdots, x_n)$ [**?**, Tag 030O]. Let us recall that if k is a perfect field, then function fields over them may not be perfect anymore: example is $\mathbb{F}_p(t)$ the function field of $\mathbb{A}^1$. However, since $\mathbb{F}_p$ is perfect, function fields are still separable over $\mathbb{F}_p$. To spell this out: an extension $\mathrm{K}/\mathbb{F}_p(t)$ may not be separable since $\mathbb{F}_p(t)$ is not perfect, but $\mathrm{K}/\mathbb{F}_p$ is still separable. In any case, the point here is that separability is relative; a proof that $\mathbb{F}_p$ is relative shows that this notion is related to being reduced [**?**, Tag030W].

$\operatorname{Spec} \mathrm{B} \hookrightarrow \operatorname{Spec} \mathrm{A}$ is that $\operatorname{Spec} \mathrm{A}$ is a thickening of B. We say that a morphism $f : \mathrm{X} \to \mathrm{Y}$ is **formally smooth** if given a solid commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathrm{B} \arrow[r] \arrow[d, hook] & \mathrm{X} \arrow[d] \\
\operatorname{Spec} \mathrm{A} \arrow[r] \arrow[ur, dashed] & \mathrm{Y},
\end{tikzcd}$$

the displayed dashed arrows exists. This is a rather abstract formulation of the notion of smoothness but is as good as the others: a scheme X over a field k is smooth if and only if X is finite type over k and $\mathrm{X} \to \operatorname{Spec} k$ is formally smooth. The latter means that for any $\mathrm{A} \to \mathrm{B}$ which is a first order thickening of k-algebras, the map of sets $\mathrm{X(A)} \to \mathrm{X(B)}$ is surjective. In the main text, we will use this to show smoothness of some groups unconditionally.

Exercise 3.2.10. *Suppose* X *is a scheme over* k *and* x *is a* k*-point of* X. *Show that the set* $\mathrm{T}_x\mathrm{X}$ *of* $k[\epsilon]/\epsilon^2$*-points of* X *whose underlying* k*-point is* x *has a natural* k*-vector space structure. Show that the tangent space of* X *at* x *(i.e.* $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$*) is naturally isomorphic to* $\mathrm{T}_x\mathrm{X}$.

4. Preschemes

The following definition is non-standard.

Definition 4.0.1. A **prescheme** is a presheaf F of affine schemes which satisfies Zariski descent. In other words, for every scheme S and every open cover U_α of S the following is an equalizer diagram.

$$\mathrm{Hom(S, F)} \to \prod_\alpha \mathrm{Hom}(\mathrm{U}_\alpha, \mathrm{F}) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Hom}(\mathrm{U}_\alpha \cap \mathrm{U}_\beta, \mathrm{F})$$

The following lemma shows that the definition of a prescheme is minimally reasonable.

Lemma 4.0.2. *Every scheme is a prescheme.*

Proof. Omitted. □

Our goal now is to show that GL(V) is a prescheme, even when V is infinite-dimensional. To do this we'll first need to show that End(V) is a prescheme, and to do this we'll show that End(V) can be written as a limit of colimits of schemes. Our goal now is to show that the property of being a prescheme is preserved by taking limits and certain colimits. The next lemma shows that to check if a presheaf is a prescheme it suffices to reduce to the case of a finite cover by affine schemes. This will be useful in proving Lemma **??** (that preschemes are preserved by filtered colimits).

Lemma 4.0.3. *The following are equivalent for a presheaf* F *of affine schemes:*

(1) F *is a prescheme.*
(2) F *satisfies descent for every affine scheme* $\operatorname{Spec} \mathrm{A}$ *and every finite open cover* $\{\operatorname{Spec} \mathrm{A}_i\}$ *of* $\operatorname{Spec} \mathrm{A}$ *by affines.*

Proof. Omitted. □

Lemma 4.0.4. *A filtered colimit (taken in presheaves) of preschemes is a prescheme.*

Proof. Let $\mathrm{X} = \operatorname{colim} \mathrm{X}_\alpha$ be a colimit of preschemes. Ultimately, the fact that X is a prescheme follows from the fact that filtered colimits commute with finite limits in **Set**. First, by Lemma **??** it suffices to check descent for a cover of $\operatorname{Spec} \mathrm{A}$ by *finitely* many affines $\{\operatorname{Spec} \mathrm{A}_i\}$. Because each X_α is a prescheme we know that

$$\mathrm{Hom}(\operatorname{Spec} \mathrm{A}, \mathrm{X}_\alpha) \to \prod_i \mathrm{Hom}(\operatorname{Spec} \mathrm{A}_i, \mathrm{X}_\alpha) \rightrightarrows \prod_{i,j} \mathrm{Hom}(\operatorname{Spec} \mathrm{A}_i \cap \operatorname{Spec} \mathrm{A}_j, \mathrm{X}_\alpha)$$

is an equalizer diagram for each α. Because the colimit defining X is filtered, it commutes with equalizers in sets, so

$$\operatorname*{colim}_{\alpha} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{X}_\alpha) \to \operatorname*{colim}_{\alpha} \prod_i \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_i, \mathrm{X}_\alpha) \rightrightarrows \operatorname*{colim}_{\alpha} \prod_{i,j} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_i \cap \mathrm{Spec}\,\mathrm{A}_j, \mathrm{X}_\alpha)$$

is also an equalizer diagram in **Set**. Because the cover $\{\mathrm{Spec}\,\mathrm{A}_i\}$ is finite, we can commute colim_α past the products to see that

$$\operatorname*{colim}_{\alpha} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{X}_\alpha) \to \prod_i \operatorname*{colim}_{\alpha} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_i, \mathrm{X}_\alpha) \rightrightarrows \prod_{i,j} \operatorname*{colim}_{\alpha} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_i \cap \mathrm{Spec}\,\mathrm{A}_j, \mathrm{X}_\alpha).$$

By the definition of the colimit in presheaves, for every affine scheme $(\mathrm{colim}\,\mathrm{X}_\alpha)(\mathrm{Spec}\,\mathrm{A}) = \mathrm{colim}\,\mathrm{X}_\alpha(\mathrm{Spec}\,\mathrm{A}_i)$, so we see that

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{X}) \to \prod_i \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_i, \mathrm{X}) \rightrightarrows \prod_{i,j} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_i \cap \mathrm{Spec}\,\mathrm{A}_j, \mathrm{X})$$

is an equalizer diagram, as desired. ☐

Lemma 4.0.5. *Any limit (taken in presheaves) of preschemes is a prescheme.*

Proof. The argument is like the one above, using the fact that limits commute. ☐

Now let's define the schemes that End(V) is built out of.

Definition 4.0.6. Let V be a finite-dimensional k-vector space. Then the vector scheme $\mathbf{V}$ has R-points

$$\mathbf{V}(\mathrm{R}) = \mathrm{V} \otimes_k \mathrm{R}$$

Lemma 4.0.7. *As above,* $\mathbf{V}$ *is really a scheme (in fact it is affine).*

Proof. Since V is finite-dimensional,

$$\mathrm{V} \otimes_k \mathrm{R} \cong \mathrm{Hom}_{\mathrm{Vect}_k}(\mathrm{V}^\vee, \mathrm{R})$$

and by the universal property of the symmetric algebra

$$\mathrm{Hom}_{\mathrm{Vect}_k}(\mathrm{V}^\vee, \mathrm{R}) \cong \mathrm{Hom}_{\mathrm{Alg}_k}(\mathrm{Sym}^*\,\mathrm{V}^\vee, \mathrm{R}).$$

Both isomorphisms above are natural in R so $\mathbf{V} = \mathrm{Spec}(\mathrm{Sym}^*\,\mathrm{V}^\vee)$. ☐

Definition 4.0.8. Let V and V′ be vector spaces. Then $\mathrm{Hom}(\mathrm{V}, \mathrm{V}')$ is the functor whose R points are

$$\mathrm{Hom}(\mathrm{V}, \mathrm{V}')(\mathrm{R}) = \mathrm{Hom}_{\mathbf{Mod}_\mathrm{R}}(\mathrm{V} \otimes \mathrm{R}, \mathrm{V}' \otimes \mathrm{R})$$

Proposition 4.0.9. $\mathrm{Hom}(\mathrm{V}, \mathrm{V}')$ *is a prescheme for any vector spaces* V, V'.

Proof. First suppose that both V and V′ are finite-dimensional. Then

$$\begin{aligned}
\mathrm{Hom}(\mathrm{V}, \mathrm{V}')(\mathrm{R}) &= \mathrm{Hom}_{\mathbf{Mod}_\mathrm{R}}(\mathrm{V} \otimes \mathrm{R}, \mathrm{V}' \otimes \mathrm{R}) \\
&= \mathrm{Hom}_{\mathrm{Vect}_k}(\mathrm{V}, \mathrm{V}' \otimes \mathrm{R}) \\
&= \mathrm{Hom}_{\mathrm{Vect}_k}(\mathrm{V}, \mathrm{V}') \otimes \mathrm{R} \\
&= \underline{\mathrm{Hom}}_{\mathrm{Vect}_k}(\mathrm{V}, \mathrm{V}')(\mathrm{R})
\end{aligned}$$

where we write $\underline{\mathrm{Hom}}_{\mathrm{Vect}_k}(\mathrm{V}, \mathrm{V}')$ for the vector scheme associated to the finite dimensional vector space $\mathrm{Hom}_{\mathrm{Vect}_k}(\mathrm{V}, \mathrm{V}')$.

Now suppose that V is finite-dimensional but let V′ be arbitrary. Write $\mathrm{V}' = \mathrm{colim}_\alpha\,\mathrm{V}'_\alpha$ where the colimit is taken over all finite-dimensional subspaces of V′. Then as before

$$\mathrm{Hom}(\mathrm{V}, \mathrm{V}')(\mathrm{R}) = \mathrm{Hom}_{\mathrm{Vect}_k}(\mathrm{V}, \mathrm{V}') \otimes \mathrm{R}$$

Since V is finite-dimensional, every map from V to V' factors through some finite dimensional V'_α, so

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Vect}_k}(V, V') \otimes R &= (\operatorname*{colim}_\alpha \mathrm{Hom}_{\mathrm{Vect}_k}(V, V'_\alpha)) \otimes R \\ &= \operatorname*{colim}_\alpha (\mathrm{Hom}_{\mathrm{Vect}_k}(V, V'_\alpha) \otimes R) \\ &= \operatorname*{colim}_\alpha \mathrm{Hom}(V, V'_\alpha)(R).\end{aligned}$$

This shows that $\mathrm{Hom}(V, V') = \mathrm{colim}_\alpha \mathrm{Hom}(V, V'_\alpha)$ is a filtered colimit of preschemes, so by Lemma **??** we know that $\mathrm{Hom}(V, V')$ is a prescheme.

Now suppose that both V and V' are arbitrary. Let $V = \mathrm{colim}_\alpha V_\alpha$ where the colimit is taken over all finite-dimensional subpsaces of V. Then

$$\begin{aligned}\mathrm{Hom}(V, V')(R) &= \mathrm{Hom}_{\mathrm{Vect}_k}(V, V' \otimes R) \\ &= \lim_\alpha \mathrm{Hom}_{\mathrm{Vect}_k}(V_\alpha, V' \otimes R) \\ &= \lim_\alpha \mathrm{Hom}(V_\alpha, V')(R).\end{aligned}$$

This shows that $\mathrm{Hom}(V, V') = \lim_\alpha \mathrm{Hom}(V_\alpha, V')$ is a limit of preschemes, so by Lemma **??** we know that $\mathrm{Hom}(V, V')$ is a prescheme. □

Corollary 4.0.10. End(V) *is a prescheme for any (possibly infinite-dimensional) vector space* V.

Proposition 4.0.11. GL(V) *is a prescheme for any (possibly infinite-dimensional) vector space* V.

Proof. Let Spec A be an affine scheme with a finite affine cover $\{\mathrm{Spec}\, A_i\}$. If f, g are two elements of GL(V)(A) that agree after restricting to each $\mathrm{Spec}\, A_i$, then f and g are equal as elements of End(V) by Corollary **??**, which implies that they're equal as $\mathrm{GL}(V)(A) \to \mathrm{End}(V)(A)$ is injective.

If $\{f_i\}$ is a collection of elements of $\mathrm{GL}(V)(A_i)$ that agree on intersections then they glue to an element $f \in \mathrm{End}(V)(A)$. Additionally, $\{f_i^{-1}\}$ is another collection of elements of $\mathrm{GL}(V)(A_i)$ that agree on intersections, so they glue to an element $g \in \mathrm{End}(V)(A)$. We claim that $g = f^{-1}$, in other words that $g \circ f = f \circ g = id$. This follows because restriction respects composition, and it suffices to check these equalities after restricting to each $\mathrm{Spec}\, A_i$. □

5. Faithfully flat descent

We discuss the idea of faithfully flat descent.

Definition 5.0.1. An R-module N is **flat** if the functor $N \otimes_R -$ is left exact (it is, of course, always right exact). Equivalently, $\mathrm{Tor}_1^R(M, N) = 0$ for any R-module M. A morphism of rings $R \to S$ is **flat** if S is flat as an R-module.

One way to think about how flat modules look like is Lazard's theorem:

Theorem 5.0.2. [**?**, Tag 058G] *An* R*-module* M *is flat if and only if* M *can be written as a filtered colimit of finitely generated free* R*-modules.*

The notion of flat morphism of schemes is ubiquitious in algebraic geometry:

Definition 5.0.3. A morphism of schemes $f : X \to Y$ is said to be **flat** if for each $x \in X$, we have that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module. More generally, a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be f**-flat** if $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,f(x)}$-module.

Flat morphisms have the expected permanence properties: they are closed under base change and compositions. Flatness can be checked on an open affine cover on the base in the following sense.

Lemma 5.0.4. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of schemes, then the following are equivalent:*

1. *f is a flat morphism;*
2. *there is an open affine covering $\mathrm{Y} = \bigcup_{i \in \mathrm{I}} \operatorname{Spec} \mathrm{A}_i$ and an open covering $f^{-1}(\operatorname{Spec} \mathrm{A}_i) = \bigcup_{j \in \mathrm{J}} \operatorname{Spec} \mathrm{B}_{ij}$ such that each $\mathrm{A}_i \to \mathrm{B}_{ij}$ is a flat morphism.*
3. *for each $y \in \mathrm{Y}$, the pullback $\mathrm{X} \times_{\mathrm{Y}} \operatorname{Spec} \mathcal{O}_{\mathrm{Y},y} \to \operatorname{Spec} \mathcal{O}_{\mathrm{Y},y}$ is flat.*

The following concept is key notion all over algebraic geometry.

Definition 5.0.5. A morphism of rings $\mathrm{R} \to \mathrm{S}$ is **faithfully flat** if it is flat and any of the following equivalent conditions hold:

1. if an R-module M is nonzero, $\mathrm{M} \otimes_{\mathrm{R}} \mathrm{S}$ is nonzero;
2. if a sequence $\mathrm{M}' \to \mathrm{M} \to \mathrm{M}''$ is not exact then neither it is exact after tensoring with S.
3. $\operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{A}$ is surjective.

For a reference about the equivalence of the above conditions see [].

Example 5.0.6. Here are some examples of flat morphisms:

1. if k is a field, any morphism $\mathrm{X} \to \operatorname{Spec} k$ is automatically flat;
2. open immersions $\mathrm{U} \subset \mathrm{X}$ are flat
3. for any scheme X, the projection morphism $\mathbb{A}^r_{\mathrm{X}} \to \mathrm{X}$ is flat since the morphsim $\mathrm{R} \to \mathrm{R}[\mathrm{T}_1, \cdots, \mathrm{T}_r]$ is flat always;
4. the projection map $\mathbb{P}^r_{\mathrm{X}} \to \mathrm{X}$ is flat;
5. in general, smooth morphisms are flat.

Note that examples 1), 3) and 4) are necessarily faithfully flat morphisms while open immersions are not necessarily faithfully flat.

Example 5.0.7. A closed immersion is flat and finitely presented if and only if it is an open immersion [**?**, Tag 0819]; hence closed immersions do not tend to be flat.

The following lemma is often called “faithfully flat descent.” The proof itself is really quite simple but is one of the key foundational pieces of modern algebraic geometry.

Lemma 5.0.8. *For any faithfully flat morphism $\mathrm{R} \to \mathrm{S}$, we have an equalizer diagram of rings*

$$\mathrm{R} \to \mathrm{S} \rightrightarrows \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}.$$

Proof. First, since being an equalizer diagram in the category of rings is the same thing as being an equalizer diagram in the category of R-modules, we are asking that the following sequence of R-modules is exact

$$0 \to \mathrm{R} \to \mathrm{S} \xrightarrow{s \mapsto s \otimes 1 - 1 \otimes s} \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}$$

Now, we note that a morphism $\mathrm{R} \to \mathrm{S}$ is faithfully flat if and only if

First, assume that the map $\mathrm{R} \to \mathrm{S}$ has a a section (of R-modules). Then □

Corollary 5.0.9. *Let k be a base ring and X be any k-scheme. Suppose that $\mathrm{R} \to \mathrm{S}$ is a faithfully flat map of k-algebras. Then we have an equalizer diagram of sets:*

$$\mathrm{X}(\mathrm{R}) \to \mathrm{X}(\mathrm{S}) \rightrightarrows \mathrm{X}(\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}).$$

5.1. Abstract descent theory.

We now want to explain the idea of abstract descent which is so crucial in mathematics. It will be useful, however, to see an instance of this in a very concrete situation (which will be used in the main body of the text).

Definition 5.1.1. Let L be a finite Galois extenson of K with Galois group G. If V is an L-vector space, then a **semilinear G-structure** on V is a G-action on the underlying additive abelian group of V such that

$$\sigma(\ell v) = \sigma(\ell)\sigma(v) \forall \sigma \in \mathrm{G}.$$

The category of L-vector spaces with a semilinear G-action will be denoted by $\mathrm{Vect}_{\mathrm{L}}^{\mathrm{G}}$. The functor $\mathrm{Vect}_{\mathrm{K}} \to \mathrm{Vect}_{\mathrm{L}}$, in fact, factors through

$$\mathrm{Vect}_{\mathrm{K}} \to \mathrm{Vect}_{\mathrm{L}}^{\mathrm{G}};$$

indeed if V is a K-vector space, then $\mathrm{V} \otimes \mathrm{L}$ admits a semilinear G-structure by declaring:

$$\sigma(v \otimes \ell) := v \otimes \sigma(\ell).$$

Proposition 5.1.2 (Descent for vector spaces). *Let* L *be a finite Galois extension of* K *with Galois group* K*, then*

$$\mathrm{Vect}_{\mathrm{K}} \to \mathrm{Vect}_{\mathrm{L}}^{\mathrm{G}},$$

is an equivalence of categories.

There are many ways to prove Proposition **??**; a slick perspective using Morita theory is given in [**?**, Appendix A.j]. But what the proposition says is quite simple: if W is an L-vector space with a semilinear G-action, then W^{G} admits naturally the structure of a K-vector space and the map $\mathrm{W}^{\mathrm{G}} \otimes_{\mathrm{L}} \mathrm{K} \to \mathrm{W}$ is actually an isomorphism.

The general pattern of descent is then as follows: we are given a ring map $f : \mathrm{R} \to \mathrm{S}$ or a scheme map $f : \mathrm{Y} \to \mathrm{X}$ and we have the tensoring functor (left adjoints)

$$f^* : \mathrm{QCoh}(\mathrm{X}) \to \mathrm{QCoh}(\mathrm{Y}) \qquad \mathrm{S} \otimes_{\mathrm{R}} - : \mathrm{Mod}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{S}};$$

This functor lands in some category of “structured S-modules”; in the example of Galois action above it’s the structure of a semilinear group action. In general it’s a **descent datum**:

Definition 5.1.3. A **descent datum** for an S-module M is an isomorphism:

$$\varphi : \mathrm{M} \otimes_{\mathrm{R}} \mathrm{S} \to \mathrm{M} \otimes_{\mathrm{R}} \mathrm{S}$$

subject to the cocycle condition

$$\begin{tikzcd}
& \mathrm{S} \otimes_{\mathrm{R}} \mathrm{M} \otimes_{\mathrm{R}} \mathrm{S} \arrow[dr, "\varphi_{12}"] & \\
\mathrm{M} \otimes_{\mathrm{R}} \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S} \arrow[ur, "\varphi_{01}"] \arrow[rr, "\varphi_{02}"] & & \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S} \otimes_{\mathrm{R}} \mathrm{M}.
\end{tikzcd}$$

where φ_{ij} is the obvious notation. Descent datum naturally forms a category $\mathrm{Desc}_{\mathrm{R}/\mathrm{S}}$; the morphisms are morphisms of S-modules which intertwines descent datum.

Example 5.1.4 (Canonical datum). Given an R-module N, we can produce the **canonical datum** on $\mathrm{M} := \mathrm{N} \otimes_{\mathrm{R}} \mathrm{S}$ as an example of descent datum via the isomorphisms (of $\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}$-modules)

$$(\mathrm{N} \otimes_{\mathrm{R}} \mathrm{S}) \otimes_{\mathrm{R}} \mathrm{S} \simeq \mathrm{N} \otimes_{\mathrm{R}} (\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}) \simeq \mathrm{S} \otimes_{\mathrm{R}} (\mathrm{N} \otimes_{\mathrm{R}} \mathrm{S}).$$

Remark 5.1.5. Definition **??** can be formulated for QCoh instead of Mod in an almost verbatim way.

This gives us a factorization of the functor $\mathrm{S} \otimes_{\mathrm{R}} - : \mathrm{Mod}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{S}}$ through

$$\mathrm{S} \otimes_{\mathrm{R}} - : \mathrm{Mod}_{\mathrm{R}} \to \mathrm{Desc}_{\mathrm{R}/\mathrm{S}};$$

In fact, $\mathrm{Desc}_{\mathrm{R}/\mathrm{S}}$ can be formulated in a Tannakian way: let $f^* := \mathrm{S} \otimes_{\mathrm{R}} -$ and f_* be the forgetful functor; then

$$f^* f_* : \mathrm{Mod}_{\mathrm{S}} \to \mathrm{Mod}_{\mathrm{S}}$$

has the structure of a comonad.

Lemma 5.1.6. *There is a canonical equivalence of categories*

$$\mathrm{Desc}_{\mathrm{R}/\mathrm{S}} \simeq \mathrm{CoMod}_{f^* f_*}(\mathrm{Mod}_{\mathrm{S}});$$

commuting over the forgetful functors to $\mathrm{Mod}_{\mathrm{S}}$*.*

In fact, if f is fatihfully flat, then the Barr-Beck theorem applies and we get

THEOREM 5.1.7 (Effectivity of descent). *There is an equivalence of categories, commuting with the functor* f^**:*

$$\mathrm{Desc}_{\mathrm{R/S}} \simeq \mathrm{Mod}_{\mathrm{R}}.$$

EXERCISE 5.1.8. *Deduce Proposition* **??** *from Theorem* **??** *by identifying the category of* G*-semilinear* L*-vector spaces with* $\mathrm{Desc}_{\mathrm{L/K}}$*.*

6. Schematic density

We are sometimes interested in the question of how $\mathrm{G}(k)$ sits in G. By Exercise 3.1.1 this is some collection of closed points of G and so we are asking for this condition in the Zariski topology. This is equivalent to the following definition:

DEFINITION 6.0.1. Let X be a finite type k-scheme; we say that a subset $\mathrm{S} \subset \mathrm{X}(k)$ is **schematically dense** if the only closed subscheme $\mathrm{Z} \hookrightarrow \mathrm{X}$ such that $\mathrm{S} \subset \mathrm{Z}(k)$ is X itself.

The key result about density is:

THEOREM 6.0.2. *Let k be an separably closed field and let* X *be a geometrically reduced k-scheme. Then* $\mathrm{X}(k)$ *is schematically dense.*

PROOF. ☐

COROLLARY 6.0.3. *Let* G *be an affine algebraic group over a field over a algebraically closed field of characteristic zero. Then* $\mathrm{G}(k)$ *is schematically dense.*

7. Grassmanians and other moduli functors

APPENDIX B

Abelian Categories

1. Definitions

DEFINITION 1.0.1. A **pointed category** is a category which has an initial object which is also a final object. We call this the **zero object**.

DEFINITION 1.0.2. For any objects X, Y in a pointed category $\mathcal{C}$, there is a distinguished element called 0 in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, which arises as the composition

$$X \to 0 \to Y$$

DEFINITION 1.0.3. For any objects X, Y in a pointed category $\mathcal{C}$, the **biproduct** of X and Y, written $X \oplus Y$ (unique if it exists) is an object equiped with maps $i_1 : X \to X \oplus Y$, $i_2 : Y \to X \oplus Y$, $p_1 : X \oplus Y \to X$, $p_2 : X \oplus Y \to Y$ so that:

- $p_1 \circ i_1 = id_X$ and $p_2 \circ i_2 = id_Y$.
- $p_2 \circ i_1 = 0$ and $p_1 \circ i_2 = 0$.
- $(X \oplus Y, i_1, i_2)$ realizes $X \oplus Y$ as the coproduct of X and Y.
- $(X \oplus Y, p_1, p_2)$ realizes $X \oplus Y$ as the product of X and Y.

LEMMA 1.0.4. *The formation of biproducts is commutative and associative.*

PROOF. Idea: follows from the corresponding properties for products and coproducts. ☐

DEFINITION 1.0.5. A **category enriched in abelian groups** is a category $\mathcal{C}$ so that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an abelian group for all X, Y, and the composition map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is a map of abelian groups.

DEFINITION 1.0.6. In any pointed category with biproducts $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ comes equipped with a natural binary operation. Namely, given $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ one can produce the map $f + g$ as the composition

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{(f,g)} Y$$

where the first map is defined because $X \oplus X = X \times X$ and the second map is defined because $X \oplus X = X \sqcup X$.

One can check that the above operation is associative, commutative, and that the map 0 is the identity element. Moreover, all of this is compatible with composition, giving any pointed category with biproducts a natural structure of a category enriched in commutative monoids.

DEFINITION 1.0.7. A pointed category with biproducts $\mathcal{C}$ is **additive** if every map $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ has an additive inverse $-f$ with respect to the addition defined above. Thus an additive category is naturally enriched in abelian groups.

DEFINITION 1.0.8. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is **additive** if it sends 0 to 0 and sends biproducts to biproducts. One can rephrase this by saying that it sends biproducts of any finite length (including length 0) to biproducts.

DEFINITION 1.0.9. In a pointed category $\mathcal{C}$, for any map $f : X \to Y$ we define the **kernel** $\ker f$ to be the equalizer of f and 0. Similarly we define the **cokernel** $\mathrm{coker} f$ to be the coequalizer of f and 0. These are unique if they exist.

Definition 1.0.10. A category $\mathcal{C}$ is **pre-abelian** if it is additive and every map has a kernel and cokernel.

Definition 1.0.11. Note that for any morphism f in any pre-abelian category $\mathcal{C}$ there is a natural map $\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$. A category is **abelian** if it is pre-abelian and this morphism is always an isomorphism. In this case we call $\ker(\operatorname{coker} f)$ the image of f (written $\operatorname{Im} f$).

Exercise 1.0.12. *Check that* $\mathrm{Rep}(\mathrm{G})$ *is abelian. In other words, verify that:*

- $\mathrm{Rep}(\mathrm{G})$ *has a zero object.*
- $\mathrm{Rep}(\mathrm{G})$ *admits biproducts.*
- *Every map in* $\mathrm{Rep}(\mathrm{G})$ *has an additive inverse.*
- *Every map has a kernel and cokernel.*
- *The natural map* $\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$ *is an isomorphism for every map* f *in* $\mathrm{Rep}(\mathrm{G})$.

Lemma 1.0.13. *A map* $f : \mathrm{X} \to \mathrm{Y}$ *in an abelian category is an isomorphism if and only if* $\ker f = \operatorname{coker} f = 0$. *Moreover,* f *is* 0 *if and only if* $\operatorname{Im} f$ *vanishes.*

Proof. Idea: If $\ker f = \operatorname{coker} f = 0$ then $\operatorname{coker}(\ker f) = \mathrm{X}$ and $\ker(\operatorname{coker} f) = \mathrm{Y}$ so f is an isomorphism. The converse is always true in a pointed category. If $\operatorname{Im} f$ vanishes then because f always factors through $\operatorname{Im} f$ we see that f must be 0. The converse holds in any pointed category (in the sense that $\operatorname{coker}(\ker f) = \ker(\operatorname{coker} f) = 0$). ☐

Definition 1.0.14. If V is an object of a category $\mathcal{C}$, a **subobject** of V is the data of an object W in $\mathcal{C}$ and a monomorphism $\mathrm{W} \hookrightarrow \mathrm{V}$.

Definition 1.0.15. Given two subobjects W, W' of V, we write $\mathrm{W} \subseteq \mathrm{W}'$ if there exists a factorization as in the diagram below:

$$\begin{tikzcd}
\mathrm{W} \arrow[r, dashed] \arrow[dr, hook] & \mathrm{W}' \arrow[d, hook] \\
 & \mathrm{V}
\end{tikzcd}$$

Note that this is a property, not additional data, because $\mathrm{W}' \hookrightarrow \mathrm{V}$ is monic, so at most one such factorization can exist. We write $\mathrm{W} = \mathrm{W}'$ if $\mathrm{W} \subseteq \mathrm{W}'$ and $\mathrm{W}' \subseteq \mathrm{W}$.

Definition 1.0.16. Suppose we have two subobjects $\mathrm{W}_1, \mathrm{W}_2$ of an object V in an abelian category $\mathcal{C}$. Let the corresponding monomorphisms be called i_1, i_2. Then we define $\mathrm{W}_1 \cap \mathrm{W}_2$ to be the kernel of $(i_1, i_2) : \mathrm{W}_1 \oplus \mathrm{W}_2 \to \mathrm{V}$ and $\mathrm{W}_1 + \mathrm{W}_2$ to be the image of (i_1, i_2). Note that $\mathrm{W}_1 \cap \mathrm{W}_2$ and $\mathrm{W}_1 + \mathrm{W}_2$ are both subobjects of V.

Exercise 1.0.17. *Show that* $\mathrm{W}_1 \cap \mathrm{W}_2$ *is the maximal subobject (with respect to the* $\subseteq$ *ordering) of* V *such that* $\mathrm{W}_1 \cap \mathrm{W}_2 \subseteq \mathrm{W}_1$ *and* $\mathrm{W}_1 \cap \mathrm{W}_2 \subseteq \mathrm{W}_2$. *Show that* $\mathrm{W}_1 + \mathrm{W}_2$ *is the minimal subobject such that* $\mathrm{W}_1 \subseteq \mathrm{W}_1 + \mathrm{W}_2$ *and* $\mathrm{W}_2 \subseteq \mathrm{W}_1 + \mathrm{W}_2$.

Definition 1.0.18. An abelian category satisfies AB3 if it admits arbitrary coproducts. An abelian category satisfies AB5 if it satisfies AB3 and the formation of filtered colimits is exact.

Exercise 1.0.19. *If* $\mathcal{C}$ *is an abelian category satisfying AB3 then* $\mathcal{C}$ *admits all colimits.*

Definition 1.0.20. If $\mathcal{C}$ is an abelian category and $\{\mathrm{W}_i\}$ is a (possibly infinite) collection of subobjects of an object V, then we write

$$\sum_{i \in \mathrm{I}} \mathrm{W}_i$$

for the minimal subobject W of V (with respect to $\subseteq$) such that $\mathrm{W}_i \subseteq \mathrm{W}$ for all $i \in \mathrm{I}$.

Exercise 1.0.21. *If $\mathcal{C}$ satisfies AB3 then*

$$\sum_{i\in \mathrm{I}} \mathrm{W}_i$$

always exists. [Hint: it is the image of $\bigoplus_i \mathrm{W}_i \to \mathrm{V}$]

Exercise 1.0.22. *If $\mathcal{C}$ satisfies AB5 then*

$$\mathrm{W}' \cap \sum_{i\in \mathrm{I}} \mathrm{W}_i = \sum_{i\in \mathrm{I}} \mathrm{W}' \cap \mathrm{W}_i$$

Exercise 1.0.23. *Check that* Rep(G) *satisfies AB5.*

2. Semisimplicity

Definition 2.0.1. An object V of an abelian category $\mathcal{C}$ is **simple** if every subobject of V is either 0 or V itself.

Definition 2.0.2. An object V of an abelian category $\mathcal{C}$ is **semisimple** if it can be written as $\bigoplus_{i\in \mathrm{I}} \mathrm{V}_i$ with each V_i simple. Note that I is allowed to be infinite, in which case $\bigoplus$ refers to the coproduct in $\mathcal{C}$.

Lemma 2.0.3. *Let $\mathcal{C}$ be an abelian category that satisfies AB5. Let $\mathrm{V} \in \mathcal{C}$ be semisimple. Then every subobject* W *of* V *has a complement* W'*–in other words* $\mathrm{V} \cong \mathrm{W} \oplus \mathrm{W}'$.

Proof. Let $\mathrm{V} = \bigoplus_{i\in \mathrm{I}} \mathrm{V}_i$ be a direct sum decomposition of V into simples. Consider the set S of $\mathrm{J} \subseteq \mathrm{I}$ so that $\mathrm{W} \cap \bigoplus_{j\in \mathrm{J}} \mathrm{V}_j = 0$. Ordering S by subset inclusion, we see that S is closed under increasing chains by Exercise **??**. By Zorn's lemma we can find a maximal J with $\mathrm{W} \cap \bigoplus_{j\in \mathrm{J}} \mathrm{V}_j = 0$. Let $\mathrm{V}' = \mathrm{W} \oplus \bigoplus_{j\in \mathrm{J}} \mathrm{V}_j$. If $\mathrm{V} \neq \mathrm{V}'$ then we can find some $\mathrm{V}_i \not\subseteq \mathrm{V}'$. Since V_i is simple, this implies that $\mathrm{V}_i \cap \mathrm{V}' = 0$, so $(\bigoplus_{j\in \mathrm{J}} \mathrm{V}_j) \oplus \mathrm{V}_i$ doesn't intersect W, contradicting the maximality of J. Thus $\mathrm{V} = \mathrm{V}'$ and we see that $\bigoplus_{j\in \mathrm{J}} \mathrm{V}_j$ is a complement of W. ☐

Corollary 2.0.4. *If* $\mathrm{V} \in \mathcal{C}$ *is semisimple then every subobject* W *of* V *is semisimple.*

Proof. If J is as in the proof of Lemma **??** then $\mathrm{W} \cong \bigoplus_{i\in \mathrm{I}\smallsetminus \mathrm{J}} \mathrm{V}_i$ via the projection away from the complement W'. ☐

Lemma 2.0.5. *Let* G *be an affine algebraic group. Then every object* $\mathrm{V} \in \mathrm{Rep(G)}$ *has a simple subobject.*

Proof. If V is finite-dimensional then this can be seen by induction on dimension. But any V is locally finite by Lemma 1.1.5. ☐

Bibliography

[Mac71] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York-Berlin, 1971

[Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017, `https://doi.org/10.1017/9781316711736`, The theory of group schemes of finite type over a field

[Stacks] The Stacks Project Authors, *The Stacks Project*, 2017, `http://stacks.math.columbia.edu`