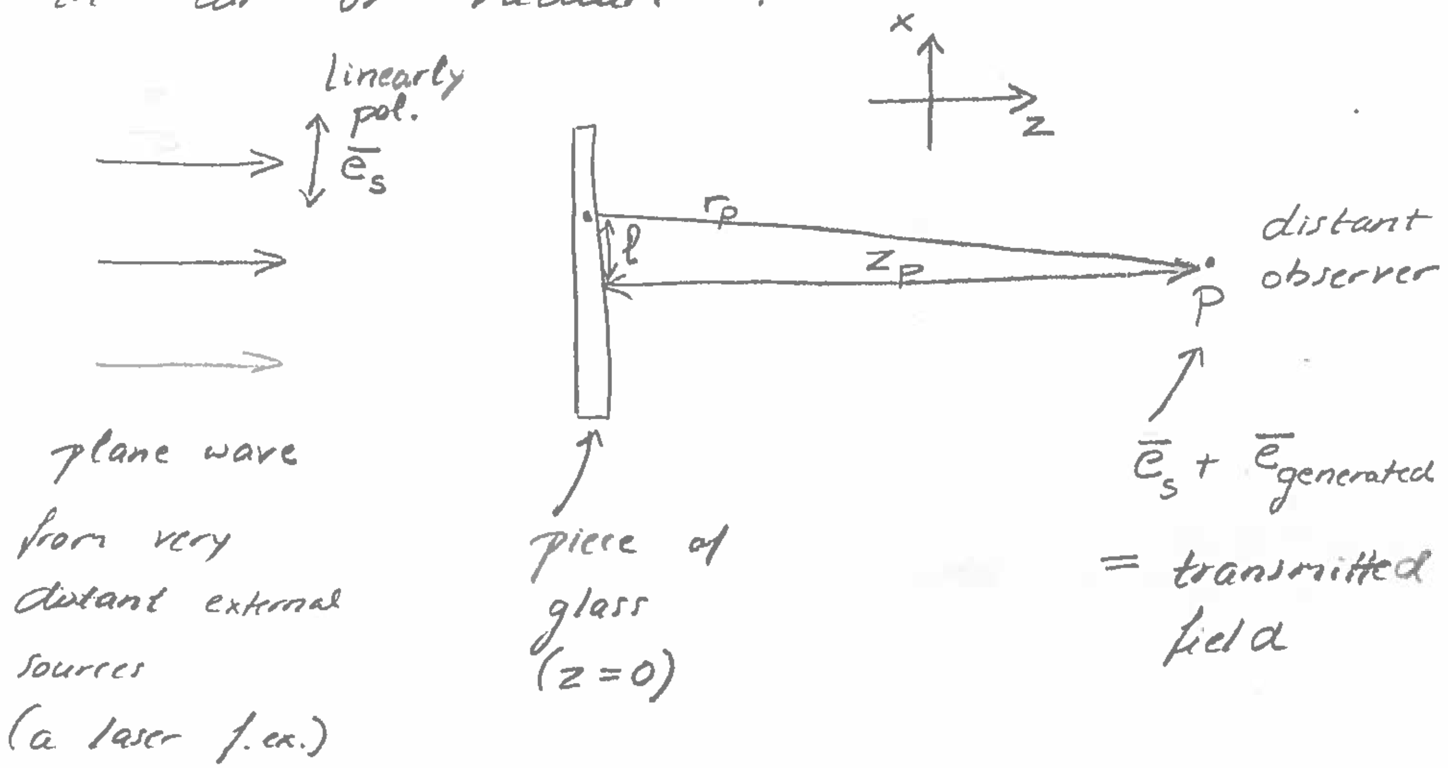
This is a good time to ask:

What is the $\boxed{\text{PHYSICS}}$ in all of this?

Q1: What is $\boxed{\text{REFRACTION}}$ really? Or:

Why does light go slower in glass than in air or vacuum?



Here we use the microscopic Maxwell-Lorentz eqs:

ALL charges ALSO from medium

$$\nabla \times \bar{b} = \mu_0 \bar{j} + \frac{1}{c^2}\frac{\partial \bar{e}}{\partial t} \qquad \nabla \cdot \bar{e} = \frac{1}{\varepsilon_0}\rho$$

$$\nabla \times \bar{e} = -\frac{\partial \bar{b}}{\partial t} \qquad \nabla \cdot \bar{b} = 0$$

space between charges is vacuum
velocity of light = c

$\bar{F}$ = Lorentz force = $q(\bar{e} + \bar{v} \times \bar{b})$

Medium (here glass) consists of atoms or molecules; in turn: atom/molecule has electrons bound to nucleus;

More specifically, what do boundary conditions correspond to <u>physically</u>:

A real boundary is not infinitely sharp — and the boundary conditions we used

D_n, B_n continuous
E_t, H_t continuous

are really a mathematical way of hiding under the rug what's really happening.

In reality: at the not infinitely sharp boundary we induce a charge distribution:

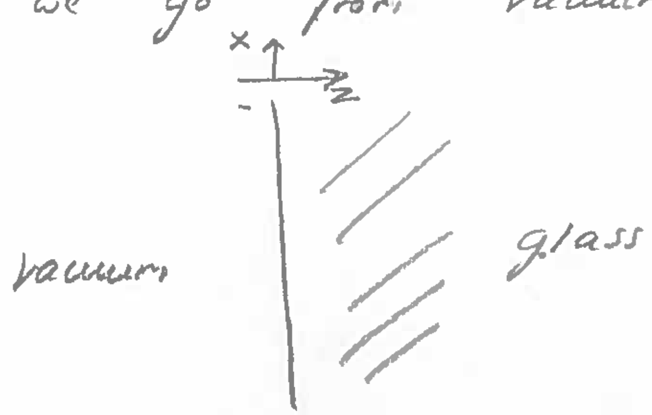
Back to microscopic Maxwell-Lorentz eqs:

F.ex.: $\nabla \cdot \bar{e} = \frac{1}{\varepsilon_0}\rho \leftarrow$ from all charges external / free + bound;

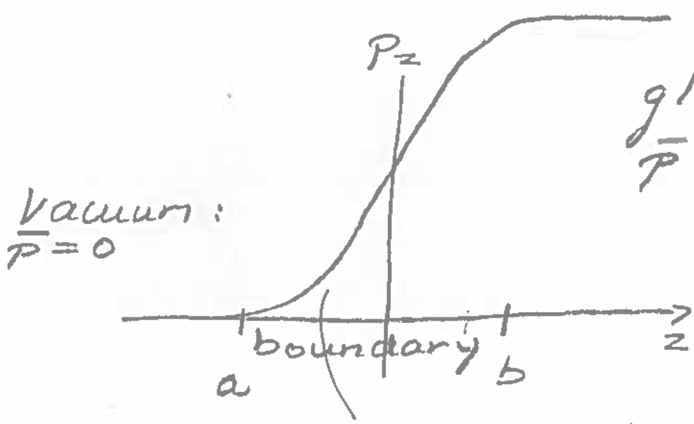
assume only bound charges here:

$\rho_b = -\nabla \cdot \bar{P}$ $\qquad \bar{P}$ = dipole density

If we go from vacuum to glass f.ex.:



What is $\bar{P}$ doing across the surface?:

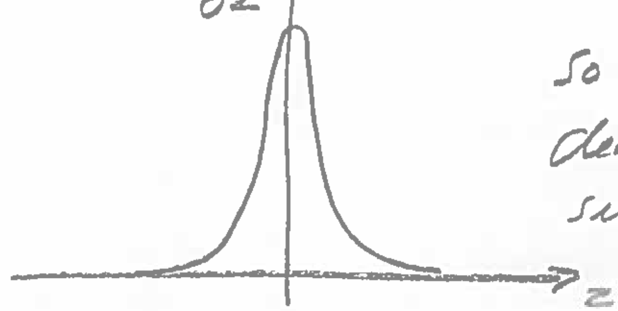


Some smooth transition at boundary

Since 'things' vary rapidly in z direction

$\nabla \cdot \bar{p} \simeq \dfrac{\partial p_z}{\partial z}$ = derivative of figure :

$\dfrac{\partial p_z}{\partial z}$ = − polarization charge density



So large charge density induced on surface

$$\nabla \cdot \bar{e} = -\frac{1}{\varepsilon_0} \nabla \cdot p = -\frac{1}{\varepsilon_0} \underbrace{\frac{\partial p_z}{\partial z}}_{\text{large}}$$

so LHS must be equally large :

$$\frac{\partial e_z}{\partial z} = -\frac{1}{\varepsilon_0} \frac{\partial p_z}{\partial z}$$

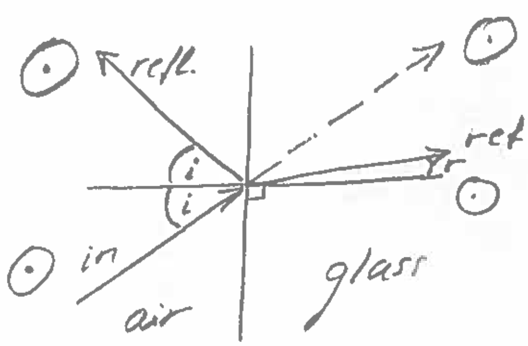
Integrate across surface :

$$\int_a^b dz \frac{\partial e_z}{\partial z} = e_z(b) - e_z(a) = -\frac{1}{\varepsilon_0}\left(p_z(b) - p_z(a)\right)$$

$\Rightarrow (\varepsilon_0 e_z + p_z)\big|_{z=b} = (\varepsilon_0 e_z + p_z)\big|_{z=a}$: d_n is continua

d_n is continuous $\Leftrightarrow$
e_z is varying drastically across the surface due to a large charge density induced at the surface.

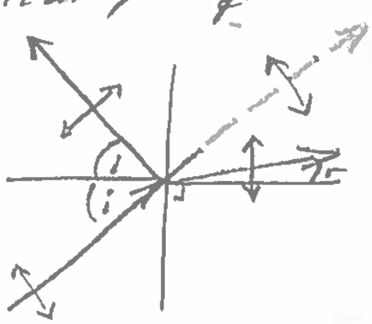
Also: we can directly relate the refracted and reflected waves to polarization of the medium.



s polarized

electrons in medium are driven into oscillation
→ emit radiation.
The radiated field from the medium cancels the incoming plane wave (dashed) inside the medium, and generates another plane wave: the refracted wave inside the medium and the reflected wave outside the medium.

Similarly for p polarized light:



Really when you think about it: the field that drives the oscillators is the refracted wave.

(1) $\nabla\cdot\bar{e} = \frac{1}{\varepsilon_0}\rho$ ← total (2) $\nabla\cdot\bar{b} = 0$

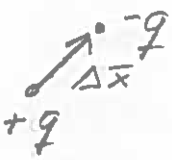
(3) $\nabla\times\bar{e} = -\frac{\partial\bar{b}}{\partial t}$ (4) $\nabla\times\bar{b} = \mu_0\bar{j} + \frac{1}{c^2}\frac{\partial\bar{e}}{\partial t}$

↑ total

Assume all charges bound (dielectric):

$\rho = -\nabla\cdot\bar{p}$, $\bar{p}$ = polarization density $= -Nq\Delta\bar{x}$

$\bar{j}$ = current density $= -Nq\frac{d(\Delta\bar{x})}{dt} = \frac{\partial\bar{p}}{\partial t}$



Solve for $\bar{e}$ and $\bar{b}$ with charge and current densities given by $\bar{p}$.... AND :

$\bar{p} = \bar{p}(\bar{e})$ gives the relation between $\bar{e}$ and $\bar{p}$. (we assume homogeneous medium).

Similarly to what we did in Lecture 1 for macroscopic Maxwell eqs :

$$\nabla\times\nabla\times\bar{e} \overset{(3)}{=} -\frac{\partial}{\partial t}\nabla\times\bar{b} \overset{(4)}{=} -\mu_0\frac{\partial^2\bar{p}}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2\bar{e}}{\partial t^2}$$

$$\| $$

$$\nabla(\nabla\cdot\bar{e}) - \nabla^2\bar{e} \underset{(1)}{=} \nabla\left(-\frac{\nabla\cdot\bar{p}}{\varepsilon_0}\right) - \nabla^2\bar{e}$$

$$\Rightarrow \nabla^2\bar{e} - \frac{1}{c^2}\frac{\partial^2\bar{e}}{\partial t^2} = -\frac{1}{\varepsilon_0}\nabla(\nabla\cdot\bar{p}) + \mu_0\frac{\partial^2\bar{p}}{\partial t^2} = -\frac{1}{\varepsilon_0}\left(\nabla(\nabla\cdot\bar{p}) - \frac{1}{c^2}\frac{\partial^2\bar{p}}{\partial t^2}\right)$$

$\frac{1}{c^2} = \mu_0\varepsilon_0$

Find plane wave solutions:

$$\bar{e}(\bar{r},t) = E_0 \bar{e}_x e^{i(kz-\omega t)}$$

= plane wave travelling in z, linearly polarized along x

Find $\omega(k) \longrightarrow n = \frac{k}{\omega} \cdot c$

For isotropic medium:

$\bar{p}$ is along $\bar{e}_x$ also:

$\nabla \cdot \bar{p} = 0$ (no variation in x).

$\bar{p}(t) \propto e^{-i\omega t}$ also (for linear medium:

$\bar{p} = \varepsilon_0 \underline{\underline{\chi}}\, \bar{e}$):

$$\left(-k^2 + \frac{1}{c^2}\omega^2\right)E_0 = -\frac{1}{\varepsilon_0}\frac{1}{c^2}\omega^2 \cdot P_0 \quad \left(\bar{p} = P_0 \bar{e}_x e^{i(k \cdot z - \omega t)}\right)$$

Now: for linear, isotropic medium:

$\bar{p}(\bar{e}) = \varepsilon_0 \chi \bar{e}$; χ scalar :

$P_0 = \varepsilon_0 \chi E_0$ so

$k^2 = \frac{\omega^2}{c^2}(1+\chi)$. We conclude:

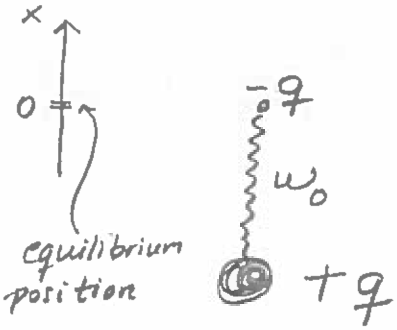
$n^2 = 1 + \chi$.

Note: $\bar{p} = N \underbrace{\alpha \varepsilon_0 \bar{e}}_{\bar{p}_{at}}$, where

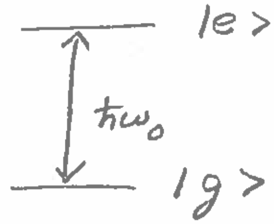
α is atomic polarizability, so

$\chi = N \cdot \alpha$

really: $\alpha = \alpha(\omega)$

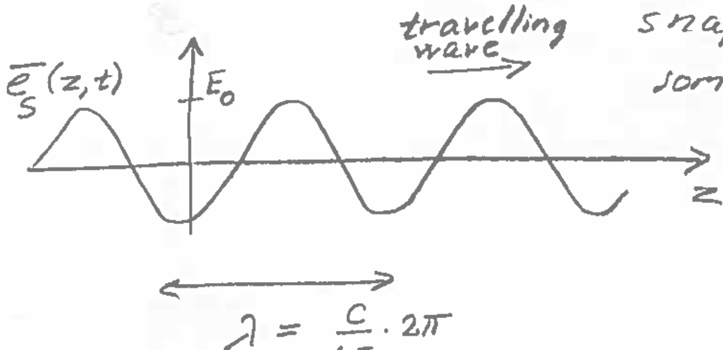


Lorentz model:
electron bound by spring



Electron driven by external field $\bar{e}_s$: at freq. ω

$$\bar{e}_s(z,t) = \bar{e}_x E_0 e^{i(kz-\omega t)}; \; k = \frac{\omega}{c}$$



snapshot at some time t

$\bar{e}_s$ linearly polarized along x f. ex.

spring — Lorentz force

$$m\ddot{x}\bar{e}_x = -m\omega_0^2 x \cdot \bar{e}_x - q\,\bar{e}_s(z_0,t)$$

position of nucleus

$x(t) = x_0 e^{-i\omega t}$ is steady state solution with

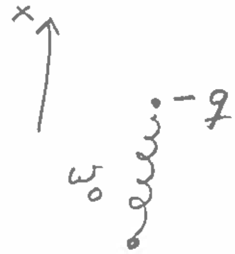
(physical solution: Re $x(t)$ really)

$$x_0 = \frac{-q E_0}{m(\omega_0^2 - \omega^2)} \qquad (z_0 = 0)$$

So we set up oscillating charge: little antenna radiates

Now find $\alpha(\omega)$:

For Lorentz model :



bound to nucleus by spring

$$x(t) = x_0 e^{-i\omega t} \quad \text{with} \quad x_0 = \frac{-q E_0}{m(\omega_0^2 - \omega^2)}$$

So :

$$\bar{P}_{at} = -q x \, \bar{e}_x = \underbrace{\frac{+\, q^2 E_0}{m(\omega_0^2-\omega^2)}}_{-q x_0} e^{-i\omega t} \, \bar{e}_x$$

$$= \frac{q^2}{m(\omega_0^2-\omega^2)} \cdot \bar{E}(z_0, t) \equiv \varepsilon_0 \alpha(\omega) \, \bar{E}(z_0, t)$$

position of nucleus

$$\boxed{\alpha(\omega) = \frac{1}{\varepsilon_0} \frac{q^2}{m(\omega_0^2-\omega^2)}}$$

Now write out n^2 :

$$n^2 = 1 + \chi = 1 + N\alpha = 1 + \frac{1}{\varepsilon_0} N \frac{q^2}{m(\omega_0^2-\omega^2)}$$

For dilute medium (N small):

$n^2 = 1 +$ "a little bit" so

$$n = \sqrt{n^2} = \sqrt{1+\chi} \simeq 1 + \tfrac{1}{2}\chi$$

However:

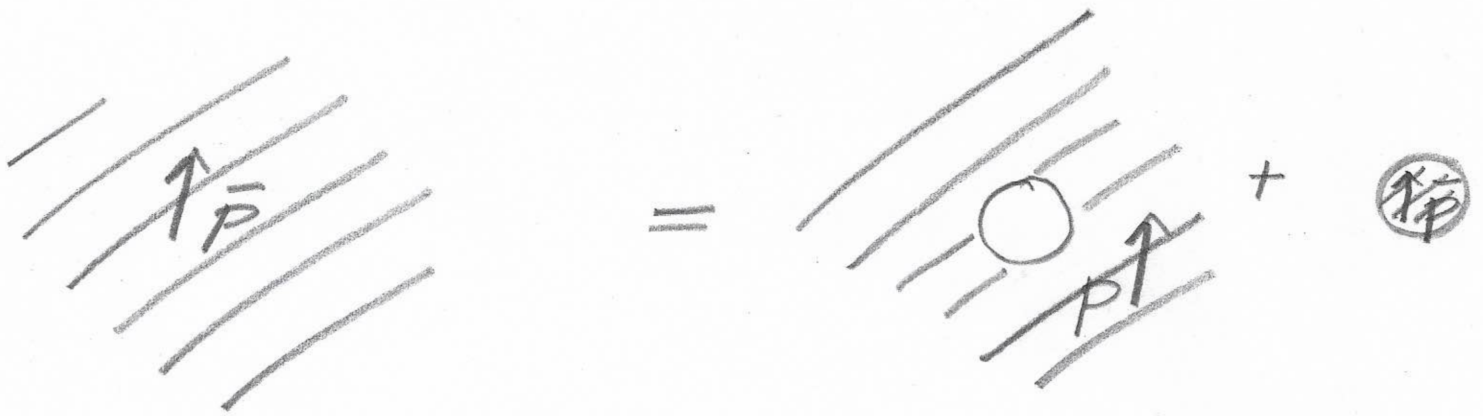
$n^2 \Rightarrow n$ (and $\frac{1}{2}$ factor) are not the only differences between dilute and dense media.

The solution $\bar{e}(\bar{r},t)$ that we just found for Maxwell eqs is really the "average" field corresponding to the average polarization density $\bar{P} = N\alpha(\omega)\cdot\varepsilon_0\cdot\bar{e}(\bar{r},t)$, and $\bar{P}$ was found by assuming that atomic polarization $\bar{P}_{at} = \varepsilon_0\alpha\,\bar{e}(\bar{r},t)$. However $\bar{P}_{at}$ should really be given as $\bar{P}_{at} = \varepsilon_0\alpha\,\bar{e}_{local}(\bar{r},t)$ with $\bar{e}_{local}(\bar{r},t)$ = the local field that is actually at the position of the atom. Clearly in general $\bar{e}_{local} \neq \bar{e}$. What do we do:

$$\bar{e}_{local} = \bar{e} + \Delta\bar{e}$$

material consists of a bunch of localized, polarized atoms represented by the polarization density $\overline{P}$:

SUPERPOSITION



Next: a localized, polarized atom placed in the void replaces the polarized sphere with volume = 1/(atom density). The atom is surrounded by neighbors represented by the polarization density $\overline{P}$.

$\overline{e}$ = the average field found from Maxwell's eqs with the polarization density $\overline{p}$ as the source.

$\overline{e}_{local}$ = the local field in the void at the atom location so $\overline{e} = \overline{e}_{local} + \overline{e}_{⊘}$,

where $\overline{e}_{⊘}$ = the electric field from and inside a uniformly polarized sphere $= -\dfrac{\overline{P}}{3\varepsilon_0}$.

So:

$$\bar{e}_{local} = \bar{e} - \bar{e}_{⊘} = \bar{e} + \underbrace{\frac{\bar{P}}{3\varepsilon_0}}^{\Delta \bar{e}}$$. So local field is enhanced compared to $\bar{e}$.

$$\bar{P} = N\overline{p_{at}} = N\alpha\varepsilon_0 \bar{e}_{local} = N\alpha\varepsilon_0\left(\bar{e} + \frac{\bar{P}}{3\varepsilon_0}\right)$$

$$\Rightarrow \bar{P} = \frac{N\alpha\varepsilon_0}{1 - \frac{N\alpha}{3}}\bar{e}.$$

So:

$$n^2 = 1 + N\alpha \longrightarrow \boxed{n^2 = 1 + \frac{N\alpha}{1 - \frac{N\alpha}{3}}}$$

$$\Longleftrightarrow \boxed{3\frac{n^2 - 1}{n^2 + 2} = N\alpha}$$

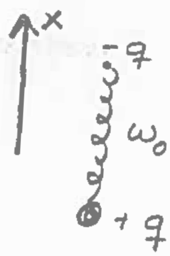
CLAUSIUS - MOSOTTI or

LORENTZ - LORENZ eq.

ASHCROFT & MERMIN Ch. 1

$\boxed{\text{CONDUCTORS}}$: conduction electron CONTRIBUTION TO susceptibility / dielectric const

So far, we have treated dielectrics – all electrons are bound (we used Lorentz model).



$$\bar{e} = E_0 \bar{\epsilon}_x e^{i(kz-\omega t)}$$

include damping term

$$m\ddot{x} = -m\omega_0^2 x - m\gamma\dot{x} - q\bar{e}(z_0,t)$$

$$x(t) = x_0 e^{-i\omega t}$$

In a conductor there are free electrons ; we can treat them exactly as we treated bound electrons : just let $\omega_0 \to 0$:

$$n^2 = 1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} = 1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{-\omega^2 - i\gamma\omega}$$

from damping

$\omega_0 = 0$ for free electrons

Note we are using the expression for n^2 <u>without</u> local field correction. Reason: conduction electrons are free to move around and sample the average electric field so that is actually the correct field for determining their response.

Electrons in a solid will collide with vibrating ion cores (phonons). Mean free time $\equiv \tau$.

After each collision the electron velocity is in a random direction (DRUDE model for free electron gas):

$$m\bar{v}_{av}(t+dt) = \underbrace{\left(1-\frac{dt}{\tau}\right)}_{\substack{\text{fraction of}\\ \text{atoms with no}\\ \text{collision in}\\ \text{interval } dt}} \cdot \left(m\bar{v}_{av}(t) - q\bar{e}(t)dt\right)$$

(contribution to $m\bar{v}_{av}(t+dt)$ from atoms colliding in dt is of second order in dt):

$$\frac{d}{dt}(m\bar{v}_{av}) = -\frac{1}{\tau}m\bar{v}_{av} - q\bar{e}(t)$$

If $\bar{e}(t)$ is <u>constant</u> <u>in</u> time interval <u>τ</u> (typ. 10^{-14}s) then a steady-state drift velocity is

$m\bar{v}_{av} = -\tau q\bar{e}$ = electric force <u>times</u> mean-free time between collisions.

Since: $\bar{j} = N\cdot\bar{v}_{av}\cdot(-q)$ we have

$$\underbrace{\bar{j} = \sigma_{DC}\cdot\bar{e}}_{\text{Ohm's law}} \quad \text{with} \quad \sigma_{DC} = \begin{Bmatrix}\text{DC}\\ \text{conductivity}\end{Bmatrix} = \frac{Nq^2}{m}\cdot\tau \equiv \sigma_0$$

τ can be obtained from measured σ_0.

This is valid for <u>Low-freq.</u> (and <u>static</u>) fields.

So: collisions give rise to a drag (effective damping) force that cancels the electric force such that the drift velocity induced by the electric field is constant in time:

$$\text{Drag force} = -\frac{1}{\tau} m \bar{v}_{av} = -m\gamma \bar{v}_{av}$$

notice $\propto -\bar{v}_{av}$ by definition of γ above.

$\gamma = \frac{1}{\tau}$ = collision rate for conduction electrons;

$$n^2 = 1 + \frac{Nq^2}{m\varepsilon_0} \frac{1}{-\omega^2 - i\omega/\tau}$$

$$= 1 + \frac{\sigma_0}{\varepsilon_0} \frac{1}{i\omega(i\omega\tau - 1)}$$

This is also valid <u>without</u> assumption of $\bar{e}(t)$ slowly varying on time scales τ: Let

$$\bar{e}(t) = \bar{E}_0 \cdot e^{-i\omega t}$$

$$\frac{d(m\bar{v}_{av})}{dt} = -\frac{m\bar{v}_{av}(t)}{\tau} - q\cdot\bar{e}(t)$$

Steady state solution: $\bar{v}_{av}(t) = \bar{v}_{av_0} e^{-i\omega t}$:

$$-i\omega m \bar{v}_{av_0} = -\frac{m\bar{v}_{av_0}}{\tau} - q\bar{E}_0 \Rightarrow \bar{v}_{av_0} = \frac{-q\bar{E}_0}{m(\frac{1}{\tau} - i\omega)}$$

$$\bar{j}(t) = -qN\bar{v}_{av}(t) = \frac{q^2 N \bar{E}_0 e^{-i\omega t}}{m(\frac{1}{\tau} - i\omega)} = \underbrace{\sigma_0 \frac{1}{(1 - i\omega\tau)}}_{\sigma(\omega)} \bar{e}(t)$$

so

$$\sigma(\omega) = \sigma_0 \frac{1}{1-i\omega\tau} \qquad (\text{note: } \sigma(\omega) \simeq \sigma_0 , \ \omega\tau \ll 1)$$

Back to microscopic Maxwell eqs – more precisely to the wave eq. with source term that we derived:

$$\nabla^2 \bar{e} - \frac{1}{c^2}\frac{\partial^2 \bar{e}}{\partial t^2} = -\frac{1}{\varepsilon_0}\left(\nabla(\nabla \cdot \bar{p}) - \frac{1}{c^2}\frac{\partial^2 \bar{p}}{\partial t^2}\right)$$

$\nabla \cdot \bar{p} = 0$ for plane prop. wave $E_0 \bar{e}_x e^{i(kz-\omega t)}$

this came from

$$-\frac{\partial}{\partial t}(\nabla \times \bar{b}) = -\frac{\partial}{\partial t}\left(\mu_0 \bar{j} + \frac{1}{c^2}\frac{\partial \bar{e}}{\partial t}\right)$$

$\bar{j}$ was set equal to $\frac{\partial \bar{p}}{\partial t}$: bound charges

NOW: $\bar{j} = \frac{\partial \bar{p}}{\partial t} + \underbrace{\sigma(\omega)\,\bar{e}}$

from free electrons

We got : $k^2 = \frac{\omega^2}{c^2}(1+\chi)$

and $n^2 = 1+\chi$

$$-\frac{\partial}{\partial t}\mu_0 \bar{j} = \underbrace{-\mu_0 \frac{\partial^2 \bar{p}}{\partial t^2}}_{\begin{array}{c}\parallel \\ +\frac{1}{\varepsilon_0}\frac{1}{c^2}\omega^2 \cdot \varepsilon_0 \chi \bar{E}_0 e^{-i\omega t}\end{array}} - \mu_0 \sigma(\omega) \bar{E}_0 e^{-i\omega t}(-i\omega)$$

$$\text{So} \quad \chi \longrightarrow \chi_{bound} + i\frac{\sigma(\omega)}{\omega \varepsilon_0}$$

and now: $\boxed{k^2 = n^2 \frac{\omega^2}{c^2}}$ with $\boxed{n^2 = 1 + \chi_{bound} + i\frac{\sigma(\omega)}{\omega\varepsilon_0}}$

remember: $n^2 = \varepsilon\mu / \varepsilon_0\mu_0 = \varepsilon/\varepsilon_0 \quad (\mu = \mu_0)$

by definition

$$n^2 = 1 + \chi_{bound} + i\frac{\sigma_0}{1-i\omega\tau}\frac{1}{\omega\varepsilon_0} \quad !$$

for dielectric (bound electrons) found earlier

from conduction electrons

Lets assume $\chi_{bound} = 0$ (we'll concentrate on free electrons)

Low frequency limit:

$\boxed{\omega\tau \ll 1}$: $n^2 \simeq 1 + i\frac{\sigma_0}{\omega\varepsilon_0} \simeq i\frac{\sigma_0}{\omega\varepsilon_0}$

$\frac{\sigma_0}{\omega\varepsilon_0} \sim 10^5$ for $\omega \sim \frac{1}{\tau}$, $\tau \sim 10^{-14}$s for Cu

So $n = \sqrt{\frac{\sigma_0}{2\varepsilon_0\omega}}(1+i)$.

The real and imaginary parts of n are equal so wave is attenuated rapidly:

The amplitude decays as $e^{-\frac{\omega}{c}\sqrt{\frac{\sigma_0}{2\varepsilon_0\omega}}z} = e^{-\sqrt{\frac{\omega\sigma_0}{2\varepsilon_0 c^2}}z}$

$= e^{-z/\delta}$ with decay length (skin depth)

$$\delta = \sqrt{\frac{2\varepsilon_0 c^2}{\omega\sigma_0}}$$

$\delta(\mathrm{Cu}) \sim 10\mu m$ at 10GHz

Waveguides for microwaves minimize losses with Au or Ag coat of walls.

High frequency limit $\boxed{\omega\tau \gg 1}$:

$$n^2 \simeq 1 - \frac{\sigma_0}{\varepsilon_0 \tau} \frac{1}{\omega^2} .$$

Insert expression for σ_0:

$$n^2 = 1 - \frac{\omega_p^{\,2}}{\omega^2} \quad ; \quad \omega_p^{\,2} = \frac{Nq^2}{m\varepsilon_0}$$

(1) n^2 is real and <u>less than 1</u>.

$\omega < \omega_p$: $n^2 < 0$ and n is imaginary so waves attenuated.

$\omega > \omega_p$: $n^2 > 0$ and n is real so waves propagate : metal transparent

Na : ω_p corresponds to a wavelength of 2100 Å (UV) so Na is transparent for wavelengths shorter than 2100 Å.

Na reflects strongly in the visible because $\omega(\text{vis}) < \omega_p$ and the refractive index has a large imag. part:

Insert a purely imaginary index into Fresnel eqs. The reflection coefficient $\left|\frac{E_0''}{E_0}\right|^2 = 1$ for $i = 90°$.

Large imaginary index also means strong absorption in the material, of course, so strongly absorbing media are highly reflective (prevents absorption really).

What is this ω_p — it is a frequency prop. to $\sqrt{\text{electron density}}$. It is called

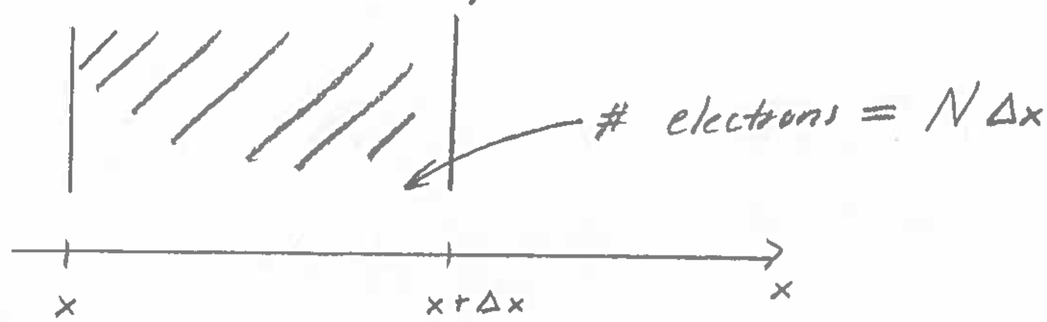
ω_p = PLASMA FREQUENCY.

what is oscillating:
It is really a many-body collective excitation of the electron gas (so quite complicated).

Electron gas uniform density in equilibrium.
(electrically neutralized by background of heavy, stationary, positively charged ion cores).

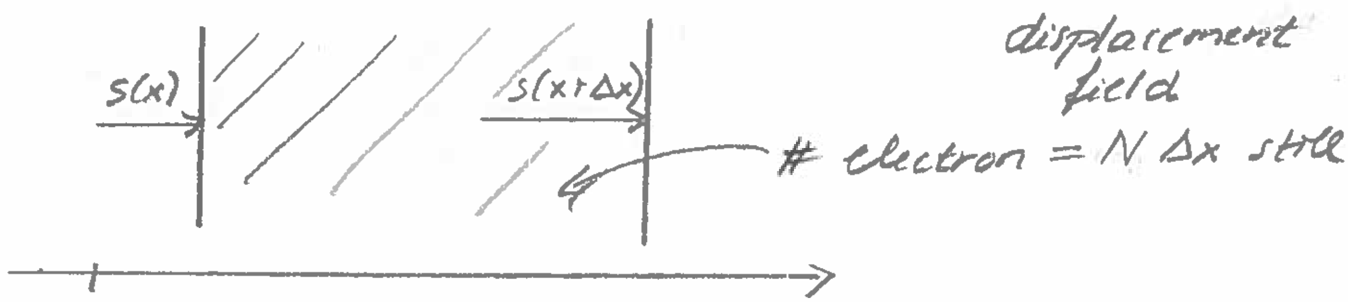
If you excite the gas:
compress the electrons somewhere f.ex.
you can excite oscillations -plasma oscillations-
of the electron gas: the e-e repulsion
causes the oscillation (gives rise to a spring constant).

In 1D: In equilibrium:



After disturbance:
Displacement of electrons at position x is $s(x)$

displacement field



$$\rho_{elect.} = -q \cdot \frac{N\Delta x}{\Delta x + \underbrace{s(x+\Delta x) - s(x)}_{\Delta s}} \simeq -qN\left(1 - \frac{\Delta s}{\Delta x}\right)$$

$$\rho_{total} = \rho_{electr} + \underset{qN}{\underset{\parallel}{\rho_{ioncores}}} = qN\frac{\Delta s}{\Delta x}$$

$$\nabla \cdot \bar{e} = \frac{\rho_{TOT}}{\varepsilon_0} = \frac{qN}{\varepsilon_0}\frac{ds}{dx} \quad \Rightarrow \quad e_x = \frac{qN}{\varepsilon_0} s(x)$$

$\nabla \cdot \bar{e} = \frac{de_x}{dx}$

e_x must be 0 when $s(x)$ is 0 so integration const $= 0$.

Furthermore:

$$m \frac{\partial^2 s(x,t)}{\partial t^2} = - q\, e_x(x,t) = -\frac{q^2 N}{\varepsilon_0} s(x,t)$$

AHA: harmonic oscillator equation:

$$s(x,t) = s_0 e^{\pm i\tilde{\omega} t}$$

$$\tilde{\omega} = \sqrt{\frac{q^2 N}{m \varepsilon_0}} = \omega_p \;!$$

Shoot keV electrons through metal foil: energy loss quantized in $\hbar\omega_p$ quanta

observed experimentally 1936 not understood – explained theoretically as late as 1953.

$\hbar\omega_p$ typically 10-15 eV.

IONOSPHERE

F layer 100 – 400 km above earth.

Ionized by rays from sun: plasma

$N = 10^4 - 10^6 \ e/cm^3$

Short waves (3-30 MHz) reflect off
$\lambda = 10 - 100\,m$

the ionosphere ($\omega < \omega_p$) and communication

between points on earth (beyond

horizon) is possible.

For communication with satellites:

$\omega > \omega_p$ desired: transparency

through ionosphere needed.