Find Real, Imaginary parts:

$$\frac{\nabla\varepsilon}{\varepsilon} = \frac{\nabla\varepsilon_1 + i\nabla\varepsilon_2}{\varepsilon_1 + i\varepsilon_2} = \frac{(\nabla\varepsilon_1 + i\nabla\varepsilon_2)(\varepsilon_1 - i\varepsilon_2)}{\varepsilon_1^2 + \varepsilon_2^2}$$

$$= \frac{\varepsilon_1\nabla\varepsilon_1 + \varepsilon_2\nabla\varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2} + i\,\frac{\varepsilon_1\nabla\varepsilon_2 - \varepsilon_2\nabla\varepsilon_1}{\varepsilon_1^2 + \varepsilon_2^2}$$

REAL PART:

$$\left[(\nabla S)^2 - \varepsilon_1\mu c^2\right]\bar{E}_0 =$$

$$= -\frac{1}{k_0}\left(\bar{E}_0 \cdot \frac{\varepsilon_1\nabla\varepsilon_2 - \varepsilon_2\nabla\varepsilon_1}{\varepsilon_1^2 + \varepsilon_2^2}\right)\nabla S + \frac{1}{k_0^2}\left[\nabla^2\bar{E}_0 + \nabla\left(\bar{E}_0 \cdot \frac{\varepsilon_1\nabla\varepsilon_1 + \varepsilon_2\nabla\varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2}\right)\right]$$

IMAGINARY PART:

$$-\varepsilon_2\mu c^2\bar{E}_0 = \frac{2}{k_0}\left[(\nabla S \cdot \nabla) + \tfrac{1}{2}\nabla^2 S\right]\bar{E}_0$$

$$+ \frac{1}{k_0}\left(\bar{E}_0 \cdot \frac{\varepsilon_1\nabla\varepsilon_1 + \varepsilon_2\nabla\varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2}\right)\nabla S$$

$$+ \frac{1}{k_0^2}\nabla\left(\bar{E}_0 \cdot \frac{\varepsilon_1\nabla\varepsilon_2 - \varepsilon_2\nabla\varepsilon_1}{\varepsilon_1^2 + \varepsilon_2^2}\right)$$

EXACT TILL NOW (assumed: $\mu(\bar{r}) = \mu$ real; $S, \bar{E}_0$ real).

$\boxed{\lambda \to 0}$ $(k_0 \to \infty)$: Use only lowest order terms:

REAL PART:

$$\left[(\nabla S)^2 - \varepsilon_1 \mu c^2\right] \bar{E}_0 = 0 \qquad (*)$$

lowest order : 0'th order

$$(\nabla S)^2 = \varepsilon_1 \mu c^2 = \frac{\varepsilon_1 \mu}{\varepsilon_0 \mu_0} \equiv \tilde{n}^2$$

S is known as EIKONAL
and (*) as EIKONAL APPROX.
BASIC EQ. OF GEOMETRIC OPTICS

$S(\vec{r}) = \text{CONST}$ defines GEOMETRICAL WAVE FRONT

(ε can be complex :)

Geometrical light rays = orthogonal trajectories to the geometrical wavefronts $S(\bar{r})$ = constants.

Oriented curves in direction of $\bar{s} = \frac{\nabla S}{|\nabla S|}$

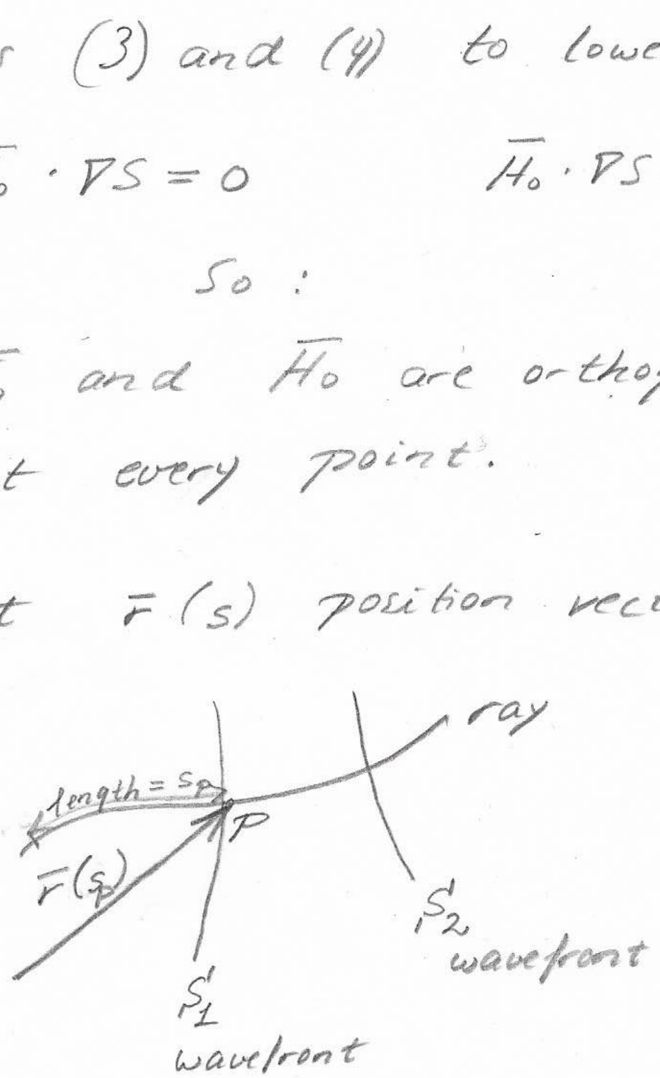
Eqs (3) and (4) to lowest order:

$\bar{E}_0 \cdot \nabla S = 0 \qquad \bar{H}_0 \cdot \nabla S = 0$

So:

E_0 and $\bar{H}_0$ are orthogonal to the ray at every point.

Let $\bar{r}(s)$ position vector of point P on ray as a function of length of arc s along the ray.



Then:

$$\bar{s} = \frac{d\bar{r}}{ds}$$

$$\frac{\nabla S}{|\nabla S|} = \bar{s} = \frac{d\bar{r}}{ds} \quad \text{and} \quad (\nabla S)^2 = \varepsilon_1 \mu c^2 = \frac{\varepsilon_1 \mu}{\varepsilon_0 \mu_0}$$

note: $\tilde{n} = n_r$ for ε real

$$\text{So}: \nabla S = \frac{d\bar{r}}{ds} \cdot |\nabla S| = \frac{d\bar{r}}{ds} \cdot \frac{\sqrt{\varepsilon_1 \mu}}{\sqrt{\varepsilon_0 \mu_0}} = \frac{d\bar{r}}{ds} \tilde{n}$$

$\boxed{\frac{d\bar{r}}{ds} \tilde{n} = \nabla S}$: Equation of Ray

Two neighboring wavefronts:

$$\mathcal{S}(\vec{r}) = S_1 \qquad \mathcal{S}(\vec{r}) = S_1 + dS.$$

$$\frac{d\mathcal{S}}{ds} = \nabla \mathcal{S} \cdot \frac{d\vec{r}}{ds} = \tilde{n} \cdot \hat{s}^2 = \tilde{n} \qquad (*)$$

$\uparrow$ $\underbrace{\text{distance along ray}}$ between wavefronts = distance along wavefront normal

DEFINE: $\int_{Ray} \tilde{n}\, ds$ = optical length along ray

$\parallel$ $\int_{Ray} n_r\, ds$ when ϵ real

Optical length between points P_1, P_2 on ray $= \int_{P_1}^{P_2} \tilde{n}\, ds \underset{(*)}{=} \int_{P_1}^{P_2} d\mathcal{S} = \mathcal{S}(P_2) - \mathcal{S}(P_1)$

For ϵ real: $c/\tilde{n} = c/n_r = v$ = phase velocity of light:

OPTICAL LENGTH EQUAL TO TIME FOR LIGHT TO TRAVEL $P_1 \to P_2$ TIMES $c \leftarrow$ vacuum speed

$$\tilde{n}\, ds = \frac{c}{v}\, ds = c\, dt \;:$$

dt: time needed for light to travel distance ds along ray

$$\boxed{\mathcal{S}(P_2) - \mathcal{S}(P_1) = \int \tilde{n}\, ds = \int_{P_1}^{P_2} c\, dt}$$

$$\frac{d\bar{r}}{ds} = \frac{\nabla S}{\tilde{n}}$$

tangent of curve (ray) $\bar{r}(s)$.

unit vector perpendicular to geometric wave front

We could solve for S and then solve this 1st order eq. for $\bar{r}(s)$.
Instead: derive 2nd order eq. for $\bar{r}(s)$:

$$\tilde{n}\frac{d\bar{r}}{ds} = \nabla S$$

$$\rightarrow \frac{d}{ds}\left(\tilde{n}\frac{d\bar{r}}{ds}\right) = \frac{d}{ds}(\nabla S) = \left(\frac{d\bar{r}}{ds}\cdot\bar{\nabla}\right)\nabla S$$

$$= \left(\frac{1}{\tilde{n}}\nabla S\cdot\bar{\nabla}\right)\nabla S$$

you check!

$$= \frac{1}{2\tilde{n}}\nabla(\nabla S\cdot\nabla S)$$

$$= \frac{1}{2\tilde{n}}\nabla(\tilde{n}^2)$$

$$= \nabla\tilde{n}$$

$$\boxed{\frac{d}{ds}\left(\tilde{n}\,\frac{d\bar{r}}{ds}\right) = \nabla\tilde{n}}$$

IN HOMOGENEOUS MEDIUM:

$$\frac{d}{ds}\left(\tilde{n}\,\frac{d\bar{r}}{ds}\right) = 0$$

$$\|$$

$$\tilde{n}\,\frac{d^2\bar{r}}{ds^2} \quad \Rightarrow \quad \frac{d^2\bar{r}}{ds^2} = 0 \quad : \quad \bar{r} = s\bar{a} + \bar{b}$$

Light rays are straight lines

Now: Equation of motion for ray:

$$\frac{d}{ds}\left(\tilde{n}\,\frac{d\bar{r}}{ds}\right) = \nabla\tilde{n}$$

$$d\underline{t} \equiv ds\,\frac{\tilde{n}}{c} \quad :$$

$$\frac{\tilde{n}}{c}\,\frac{d}{d\underline{t}}\left(\tilde{n}\,\frac{\tilde{n}}{c}\,\frac{d\bar{r}}{d\underline{t}}\right) = \nabla\tilde{n}$$

$$\frac{d}{d\underline{t}}\left(\tilde{n}^2\,\frac{d\bar{r}}{d\underline{t}}\right) = c^2\,\frac{\nabla\tilde{n}}{\tilde{n}}$$

$$\left(\nabla(\tilde{n}^2)\cdot\frac{d\bar{r}}{d\underline{t}}\right)\frac{d\bar{r}}{d\underline{t}} + \tilde{n}^2\,\frac{d^2\bar{r}}{d\underline{t}^2} = \frac{c^2}{2}\,\frac{\nabla(\tilde{n}^2)}{\tilde{n}^2}$$

$$\left(\frac{\nabla(\tilde{n}^2)}{\tilde{n}^2}\cdot\frac{d\bar{r}}{d\underline{t}}\right)\frac{d\bar{r}}{d\underline{t}} + \frac{d^2\bar{r}}{d\underline{t}^2} = \frac{c^2}{2}\,\frac{\nabla(\tilde{n}^2)}{\tilde{n}^4} \qquad (*)$$

Remember: $\tilde{n}^2 = \dfrac{\varepsilon_1 \mu}{\varepsilon_0 \mu_0}$

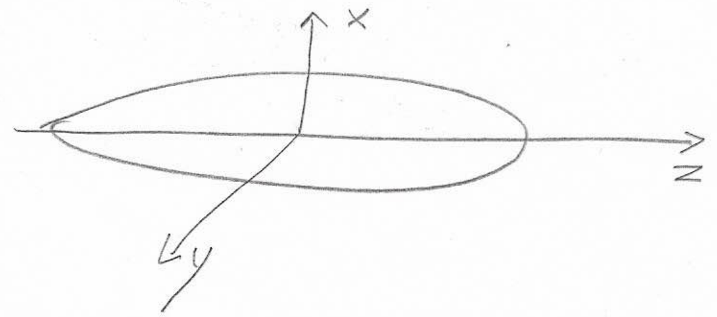
atomic density

$$\varepsilon = \varepsilon_0 (1+\chi) = \varepsilon_0 + N \alpha \varepsilon_0$$

$\varepsilon_0 \cdot$ atomic polarizability $= \dfrac{e^2}{m} \dfrac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Set $\mu = \mu_0$:

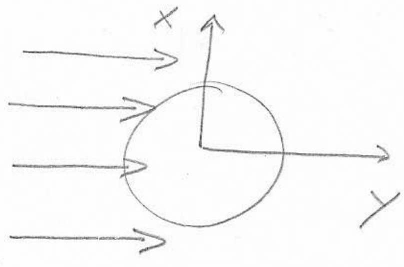
Let $z=0$: $\tilde{n}^2 = 1 + \mathrm{Re}(N\alpha) = 1 + A e^{-(x^2+y^2)/(2\sigma^2)}$



For: Non-condensed atom cloud ($\sigma^2 \propto k_B T$)

or small condensate ($\sigma = \sqrt{\frac{\hbar}{M\omega}}$ = size of ground state in harmonic oscillator)

In $z=0$ plane



probe laser beam

What happens?

$$\nabla(\tilde{n}^2) = -A \frac{\cancel{2}\,\bar{r}_\perp}{\cancel{2}\sigma^2} e^{-r_\perp^2/(2\sigma^2)}$$

where

$$\bar{r}_\perp = (x, y)$$

INTO (*):

$$\left(\frac{-A \bar{r}_\perp}{\sigma^2} \frac{e^{-(\,)}}{1+Ae^{-(\,)}} \cdot \frac{d\bar{r}_\perp}{d\underline{t}} \right) \cdot \frac{d\bar{r}_\perp}{d\underline{t}} + \frac{d^2 \bar{r}_\perp}{d\underline{t}^2}$$

$$= \frac{c^2}{2} \left(\frac{-A \bar{r}_\perp}{\sigma^2} \right) \frac{e^{-(\,)}}{(1+Ae^{-(\,)})^2} \;;$$

$$0 = \frac{d^2 \bar{r}_\perp}{d\underline{t}^2} - \frac{A}{2\sigma^2} \cdot \frac{d(r_\perp^2)}{d\underline{t}} \frac{e^{-(\,)}}{1+Ae^{-(\,)}} \frac{d\bar{r}_\perp}{d\underline{t}} + \frac{c^2 A}{2\sigma^2} \frac{e^{-(\,)}}{(1+Ae^{-(\,)})^2} \bar{r}_\perp$$

Let $\tau = \omega \underline{t}$ unitless:

$$0 = \frac{d^2 \bar{r}_\perp}{d\tau^2} - \frac{A}{2\sigma^2} \frac{d(r_\perp^2)}{d\tau} \frac{e^{-r_\perp^2/(2\sigma^2)}}{1+Ae^{-r_\perp^2/(2\sigma^2)}} \frac{d\bar{r}_\perp}{d\tau}$$

$$+ \frac{c^2}{\omega^2} \frac{A}{2\sigma^2} \frac{e^{-r_\perp^2/(2\sigma^2)}}{(1+Ae^{-r_\perp^2/(2\sigma^2)})^2} \bar{r}_\perp$$

$$= \frac{1}{k_0^2} = \left(\frac{\lambda_0}{2\pi} \right)^2$$

Put back-in $\tilde{n}^2$:

$$0 = \ddot{\bar{r}}_\perp(\tau) - \left[\frac{\tilde{n}^2 - 1}{\tilde{n}^2} \, \frac{1}{\sigma^2} \, \bar{r}_\perp(\tau) \cdot \dot{\bar{r}}_\perp(\tau) \right] \dot{\bar{r}}_\perp(\tau)$$

$$+ \frac{\lambda_0^2}{8\pi^2} \, \frac{1}{\sigma^2} \, \frac{\tilde{n}^2 - 1}{\tilde{n}^4} \, \bar{r}_\perp(\tau)$$

Equation valid only for Gaussian distrib.

Gaussian N

$\downarrow$

Parabolic distribution:

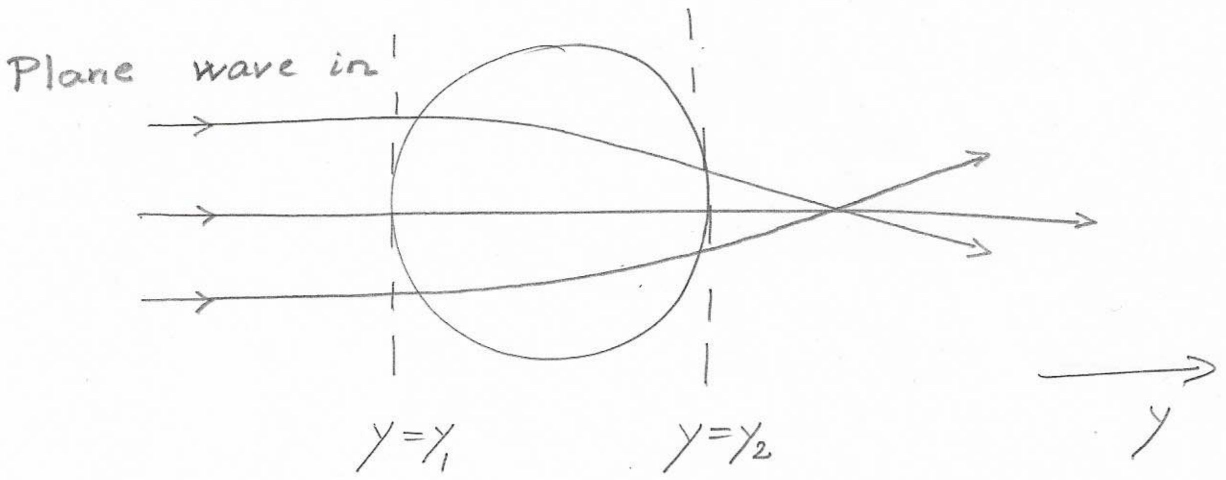
$$N = A - B(x^2 + y^2 + \varepsilon z^2)$$

for <u>large condensate</u>

Solve for $\bar{r}_\perp(\tau)$: then we have ray trajectory

$$S(P_2) - S(P_1) = \int_{P_1}^{P_2} \tilde{n}\, ds = \int_{P_1}^{P_2} c\, dt$$

$$\underbrace{e^{i k_0 S(\bar{r})} e^{-i\omega t}}_{\text{found}} \quad = \frac{c}{\omega} \int_{P_1}^{P_2} d\tau = \frac{\lambda_0}{2\pi} \overbrace{\int_{P_1}^{P_2} d\tau}^{\Delta\tau}$$



In plane $y=y_1$: $\quad \bar r_\perp(\tau) = \begin{pmatrix} x \\ y_1 \end{pmatrix}$; $\tau = 0$

Cloud is hit by plane wave:

$$\bar E_0(\bar r)\, e^{ik_0 S(\bar r)}\, e^{-i\omega t} = E_z\, e^{ik_0 y_1}\, e^{-i\omega t}$$

wavefront = plane $y = y_1$

$$\text{So : } \bar s = \frac{d\bar r}{ds} = \hat e_y$$

$$\frac{d\bar r}{c\,dt} = \frac{\omega}{c}\frac{d\bar r}{d\tau} = \frac{2\pi}{\lambda_0}\dot{\bar r}(\tau)$$

Initial condition in plane $y = y_1$: $\bar r_\perp = \begin{pmatrix} x_1 \\ y = y_1 \end{pmatrix}$ at $\tau = 0$ $\qquad \dot{\bar r}_\perp = \frac{\lambda_0}{2\pi}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Propagate each ray (defined by x_1) to plane $\underline{y = y_2}$

position

Propagate to plane $y = y_2$: gives phase

Note x_2 and $\tilde{\tau}_2$ values for each ray.

At $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ we have phase factor $\boxed{e^{ik_0 \Delta S} e^{ik_0 y_1}}$

S_1

$$\boxed{\Delta S = S_2 - S_1} = \int_{P_1}^{P_2} \tilde{n}\, ds = \frac{\lambda_0}{2\pi} \int_{P_1}^{P_2} d\tau$$

$\tau = 0$ at $y = y_1$

$$= \frac{\lambda_0}{2\pi} \tau_2 = \boxed{\frac{1}{k_0} \tau_2}$$

This defines phase distribution in plane $y = y_2$

and thereby wavefront distortion caused by atom cloud.

From here we propagate through imaging system using Fourier optics

How about $E_0(\vec{r})$: absorption

ABSORPTION:

IMAGINARY PART in nearly homogeneous medium:

$$-\varepsilon_2 \mu c^2 \bar{E}_0 = \frac{2}{k_0}(\nabla S \cdot \nabla)\bar{E}_0$$

$$= \frac{2}{k_0}\left(\tilde{n}\frac{d\bar{r}}{ds}\cdot\nabla\right)\bar{E}_0$$

Again: $d\underline{t} = ds \cdot \frac{\tilde{n}}{c}$

$$-\varepsilon_2 \mu c^2 \bar{E}_0 = \frac{2}{k_0}\frac{\tilde{n}^2}{c}\left(\frac{d\bar{r}}{d\underline{t}}\cdot\nabla\right)\bar{E}_0$$

$$-\varepsilon_2 \mu c^2 \bar{E}_0 = \frac{2}{k_0}\frac{\tilde{n}^2}{c}\frac{d\bar{E}_0}{d\underline{t}}$$

$$\Rightarrow \frac{d\bar{E}_0}{d\underline{t}} = -\frac{\varepsilon_2 \mu c^2}{2}\cdot\frac{k_0 c}{\tilde{n}^2}\bar{E}_0$$

$$\Rightarrow \frac{d\bar{E}_0}{d\tau} = -\frac{\varepsilon_2 \mu c^2}{2\tilde{n}^2}\left(\frac{k_0 c}{\omega}\right)\bar{E}_0 = -\frac{\varepsilon_2 \mu}{\varepsilon_0 \mu_0}\frac{1}{2\tilde{n}^2}\bar{E}_0$$

$\tau = \omega t$

$k_0/k_0 = 1$

$$\bar{E}_0(\tau) = \bar{E}_0(\tau=0)\, e^{-\gamma\tau} \quad ; \quad \gamma = \frac{\varepsilon_2 \mu}{\varepsilon_0 \mu_0}\frac{1}{2\tilde{n}^2}$$

if $\varepsilon_2/\varepsilon_1$ indep. of $\bar{r}$

f. ex.: HOMOGENEOUS MEDIUM

$$= \frac{\varepsilon_2 \mu}{\varepsilon_0 \mu_0}\frac{1}{2}\frac{\varepsilon_0 \mu_0}{\varepsilon_1 \mu}$$

$$= \frac{1}{2}\frac{\varepsilon_2}{\varepsilon_1}$$

If $\varepsilon_2/\varepsilon_1$ is spatially dependent we have for linearly polarized light along z:

$$\frac{dE_{0z}}{d\tau} = -\frac{1}{2}\frac{\varepsilon_2}{\varepsilon_1}E_{0z} \Rightarrow \frac{dE_{0z}}{E_{0z}} = -\frac{1}{2}\frac{\varepsilon_2}{\varepsilon_1}d\tau .$$

Then

$$\int_{P_1}^{P_2}\frac{dE_{0z}}{E_{0z}} = -\frac{1}{2}\int_{\tau_1}^{\tau_2}\frac{\varepsilon_2(\bar{r}(\tau))}{\varepsilon_1(\bar{r}(\tau))}\,d\tau, \text{ where}$$

$$\int_{P_1}^{P_2}\frac{dE_{0z}}{E_{0z}} = \ln E_{0z}(P_2) - \ln E_{0z}(P_1)$$

$$= \ln\frac{E_{0z}(P_2)}{E_{0z}(P_1)}, \text{ resulting in}$$

$$E_{0z}(P_2) = e^{-\frac{1}{2}\int_{\tau_1}^{\tau_2}d\tau\,\varepsilon_2(\bar{r}(\tau))/\varepsilon_1(\bar{r}(\tau))}E_{0z}(P_1)$$

Include this in wavefront at plane $y \hat{=} y_2$:

$$E_{0z}(P_2)\,e^{ik_0 S(\tau_2)}\,e^{-i\omega t}.$$

When is EIKONAL APPROX
RAY OPTICS
VALID ? :

$$\lambda \to 0 \quad \left(\frac{1}{k_0} \to 0\right) \quad ;$$

Need dimensionless quantity :

$$\frac{1}{k_0} \frac{\nabla \varepsilon}{\varepsilon} \ll 1$$

$$\parallel$$

$$\frac{\lambda_0}{2\pi} \frac{\nabla \varepsilon}{\varepsilon} \Rightarrow \frac{\nabla \varepsilon}{\varepsilon} \ll \frac{1}{\lambda_0}$$

$$\frac{1}{k_0} \frac{\nabla E_0}{E_0} \ll 1 \Rightarrow \frac{\nabla E_0}{E_0} \ll \frac{1}{\lambda_0}$$

cannot treat focal points