AP 217
Problem Set 2

Fall 2024

Please include any computer code you used to answer the questions. This problem is not as long as the length of this document would suggest! Most of what is contained here is necessary background information.

1 Introduction: Multi-layer Thin Film Stacks

In class, we derived the Fresnel equations for the reflection and transmission of a plane wave at a single interface (Fig. 1). However, a structure of interest may, in general, be composed of many layers. Multi-layer stacks of thin films are commonly encountered in nature (e.g., an oil slick on water) and routinely employed in optics as anti-reflection coatings and color filters. In this problem, you will develop a very general understanding of these.

Where to start? An immediately apparent strategy is to recursively trace ray paths through our N layer stack, with each ray being reflected and transmitted at every interface with the appropriate Fresnel coefficients r and t. While this yields a closed-form solution for a single layer sandwiched between two semi-infinite media, it becomes untenable for many layers.

The purpose of this problem is to familiarize you with a very general strategy known as the *transfer matrix formalism.* In what follows, **assume s polarized light**. For your reference, the Fresnel coefficients for this situation (for non-magnetic media) are:

$$r_{12} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}, \tag{1}$$

$$t_{12} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}. \tag{2}$$

Here, the subscript (1) refers to the first medium from which light is incident and (2) refers to the medium into which light is transmitted — n_1, for example, is the index of the refraction of the first medium and θ_1 is the angle of the propagation vector relative to the surface normal in that medium. Here, r_{12} is the (amplitude) reflection coefficient for light incident *from* medium 1, and t_{12} is the (amplitude) transmission coefficient.

Suppose we have N layers of material labeled 0, 1, ..., $N-1$ where layers 0 and $N-1$ are semi-infinite. Light is incident from layer 0 at an angle θ_0. At each interface, the light refracts

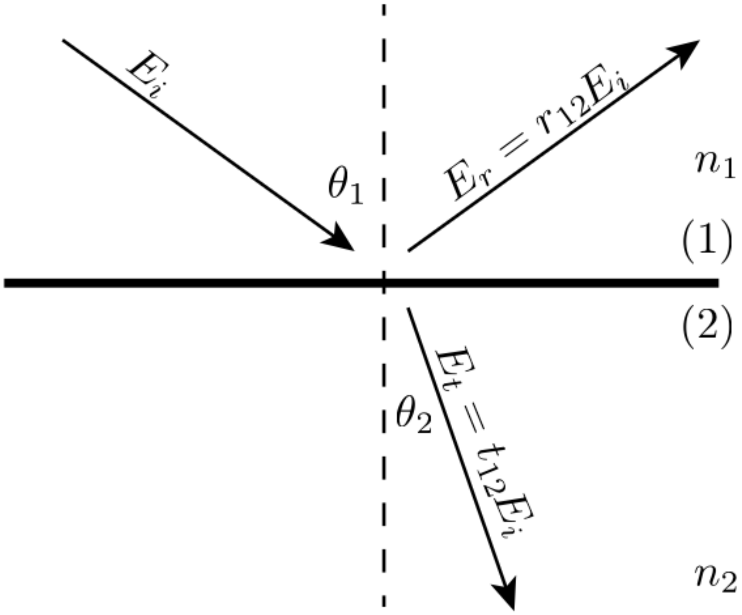

Figure 1: A plane wave with E-field amplitude E_i undergoes a single reflection and refraction at a material interface. The Fresnel coefficients are derived by appropriate matching of boundary conditions in this picture.

and so θ in each layer takes on a different value as dictated by Snell's Law, $n_i \sin\theta_i = n_j \sin\theta_j$ –stated differently, the wave-vector parallel to the layers is a conserved quantity, $k^i_\parallel = k^j_\parallel$.

There are an infinite number of forward and backward propagating rays in each layer, owing to the infinity of reflections and transmissions in the stack. Within a layer, all the forward and all the backward propagating rays propagate at a single angle so we can consider just one net forward propagating wave with amplitude v_j, and one net backward propagating wave with amplitude w_j in each layer j (Fig. 2)[1]. We assert that v_j and w_j have 0 phase at the start of layer j at its interface with layer $j-1$ (this is our phase reference). Through the layer j each picks up a phase $\delta_j = n_j k_0 \cos\theta_j d_j$, where d_j is the layer's thickness, n_j is its refractive index, θ_j is the angle of propagation in the layer measured from the interface's normal, and k_0 is the free-space wavevector $k_0 = \frac{w}{c} = \frac{2\pi}{\lambda_0}$.

If we focus on the interface between layers j and $j+1$, we observe that v_j and w_{j+1} propagate toward the interface and w_j and v_{j+1} propagate away from it. We could well start from scratch by again matching the boundary conditions from Maxwell's Equations directly. Instead, we notice that this is equivalent to a superposition of two simpler situations with which we are very familiar (Fig. 3):

(i) v_j is incident from layer j, creating a reflected wave with amplitude $w_j^{(1)}$ and a transmitted wave with amplitude $v_{j+1}^{(1)}$.

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[1] The notation in this problem was largely adapted from Byrnes, S: "Multilayer optical calculations", arXiv:1603.02720, 2016. It is an excellent source of further information.

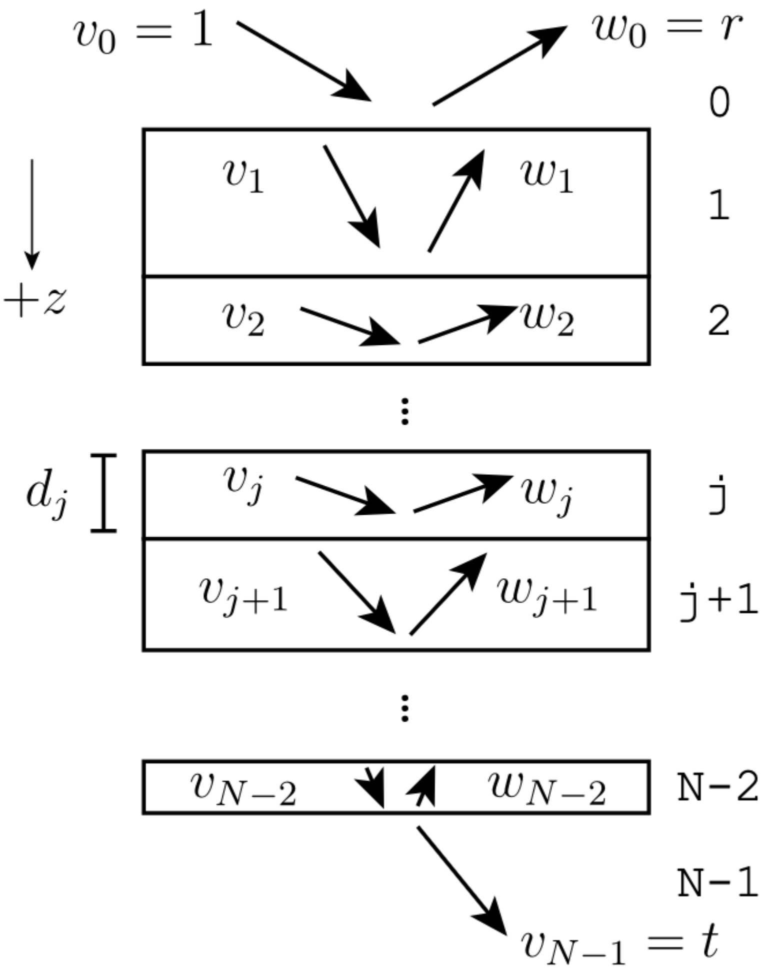


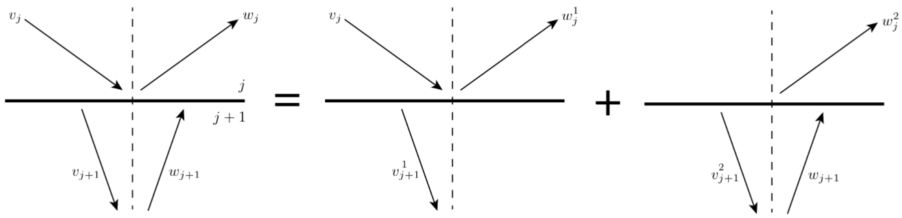
Figure 2: A plane wave of unit amplitude is incident on an N-layer stack. In each layer j, we consider just one net forward propagating wave with amplitude v_j and one net backward propagating wave with amplitude w_j; each is defined relative to the top of each layer.

(ii) w_{j+1} is incident from layer $j+1$, creating transmitted wave $w_j^{(2)}$ and reflected wave $v_{j+1}^{(2)}$.

Each of these situations is now equivalent to the simple one-interface problem for which the Fresnel coefficients were derived using the electromagnetic boundary conditions for the given angle of incidence. We can, then, use them freely here and these boundary conditions will be satisfied automatically.

2 Problems

1. The Fresnel coefficients Eqs. 1 and 2 are written in a form slightly different than derived in the course notes. Prove that the Fresnel coefficients from class are equivalent to Eqs.



$$v_{j+1} = v_{j+1}^1 + v_{j+1}^2$$
$$w_j = w_j^1 + w_j^2$$

Figure 3: The situation of forward and backward propagating plane waves at the interface of materials j and $j+1$ can be thought of as being identical to a superposition of two Fresnel-like problems, one in which a plane-wave is incident from above and another in which one is incident from below.

1 and 2. Note that the notation is slightly different than in the lecture notes.

2. Show, by superposition of the above two situations, that

$$w_j = w_j^{(1)} + w_j^{(2)} = v_j e^{2i\delta_j} r_{j,j+1} + w_{j+1} e^{i\delta_j} t_{j+1,j}$$

   and

$$v_{j+1} = v_{j+1}^{(1)} + v_{j+1}^{(2)} = v_j e^{i\delta_j} t_{j,j+1} + w_{j+1} r_{j+1,j}$$

   (don't think too hard; recall that the phase reference for v_j and w_j is at the interface between layers j and $j-1$)

3. These relations can be written in matrix form, with

$$\begin{bmatrix} v_j \\ w_j \end{bmatrix} = M_j \begin{bmatrix} v_{j+1} \\ w_{j+1} \end{bmatrix}$$

   Show that $M_j = \frac{1}{t_{j,j+1}} \begin{bmatrix} e^{-i\delta_j} & -e^{-i\delta_j} r_{j+1,j} \\ e^{i\delta_j} r_{j,j+1} & e^{i\delta_j}(t_{j,j+1}t_{j+1,j} - r_{j,j+1}r_{j+1,j}) \end{bmatrix}$.

   By inspection of the Fresnel coefficients, $r_{j,j+1} = -r_{j+1,j}$ and $t_{j+1,j}t_{j,j+1} - r_{j,j+1}r_{j+1,j} = 1$. This simplifies M_j to the form

$$M_j = \frac{1}{t_{j,j+1}} \begin{bmatrix} e^{-i\delta_j} & e^{-i\delta_j} r_{j,j+1} \\ e^{i\delta_j} r_{j,j+1} & e^{i\delta_j} \end{bmatrix}$$

4. Above, we have shown that if we know the forward and backward propagating field amplitudes in layer $j+1$, we can directly find those in layer j.

   This suggests that we work backwards in the stack, starting from the last layer (which is semi-infinite, numbered $N-1$) where we know there to be no back-reflection:

$$\begin{bmatrix} v_{N-2} \\ w_{N-2} \end{bmatrix} = M_{N-2} \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Again, we have $v_0 = 1$ and $w_0 = r$. That is (note that for $M_0, \delta_0 = 0$):

$$\begin{bmatrix} 1 \\ r \end{bmatrix} = \mathrm{M}_0 M_1 \dots M_{N-3} M_{N-2} \begin{bmatrix} t \\ 0 \end{bmatrix} = \prod_{j=0}^{N-2} M_j \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Once M is computed for the stack, we see that $t = \frac{1}{M_{11}}$ and $r = \frac{M_{21}}{M_{11}}$. Moreover, once r and t are known, we can use each layer's M_i to trace backward to find the fields in any layer.

The coefficients r and t are complex numbers, due to phase shifts light undergoes upon reflection and transmission from the stack. Often we're most interested in the power reflection and transmission coefficients, $R = |r|^2$ and $T = \frac{cos\theta_{N-1}}{cos\theta_0} \frac{n_{N-1}}{n_0} |t|^2$, where in calculating T we have remembered that intensity $I \propto n|E|^2$ and must also be multiplied by the cosine of the exit angle to accurately reflect the flux of energy of a tilted wave with respect to a plane parallel to the stack.

Let's consider a simple example stack, a thin film of diamond on a glass substrate:

Material	**Index**	**Thickness (μm)**
Air	1.0	∞
Diamond	2.41	0.200
Glass	1.45	∞

This single slab is a simple example of a **Fabry-Perot resonator**. Assume light is incident from the top material in the table.

Produce the following plots:

(i) R and T vs. λ_0 (in μm or nm), from $\lambda_0 = 0.100 - 0.600$ μm at normal incidence. Assume no chromatic dispersion in the refractive indices (assume that they are constant for all wavelengths). Note that $\Delta\lambda$ between transmission peaks (resonances) is known as the *free spectral range*. How does this plot change if instead of diamond we have a layer of silicon nitride ($n = 2.01$)? The change in the sharpness of the peaks is known as a change in the *finesse* of the cavity – a fancier way of saying quality factor.

(ii) R and T vs. incident angle θ, from $\theta = 0 - 89°$ at $\lambda_0 = 510$ nm.

5. **Dielectric Mirror.** Consider now a quarter wave stack. In a quarter wave stack, dielectric layers of high and low index alternate with each layer having quarter wavelength thickness at the wavelength of interest ($d_i = \frac{\lambda_0}{4n_i}$). For the purposes of this problem, $\lambda_0 = 510$ nm. Suppose we make a quarter wave stack from titanium dioxide ($n = 2.35$) and glass ($n = 1.45$). To be explicit:

Material	**Index**	**Thickness (μm)**
Titanium dioxide	2.35	$\frac{0.510}{2.35(4)}$
Glass	1.45	$\frac{0.510}{1.45(4)}$

When this two-layer structure repeats many times, we have a Bragg mirror, or, in the language of modern optics, a 1D photonic crystal.

Task

> On the same set of axes, plot the transmission T of the above-described quarter wave stack at normal incidence versus wavelength (from $\lambda_0 = 200$ to 1000 nm) for both i) 3 and ii) 100 repetitions of the two layers. Assume the structure is bounded by air ($n = 1$) on either side.
> This is a numerical way of probing the band structure of the 1D photonic crystal without doing much math. The wavelength ranges with near 0 transmission correspond to the photonic band gaps of the structure.