Princeton University
Spring 2025 MAT425: Measure Theory
HW1 Sample Solutions
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Problem 1. Provide an example of a sequence of Riemann integrable functions $f_n : [a,b] \to \mathbb{R}$ which converges pointwise but not uniformly to some function $\lim_n f_n : [a,b] \to \mathbb{R}$ which is also Riemann integrable, but such that

$$\lim_n \int_{[a,b]} f_n \neq \int_{[a,b]} \lim_n f_n$$

and even both quantities are finite. Conclude that

$$\lim_n \int_{[a,b]} \neq \int_{[a,b]} \lim_n$$

is violated not only because of a lack of Riemann integrability.

Solution. Consider $[a,b] = [0,1]$ and $f_n = n1_{(0,n^{-1}]}$, where $1_A(x) = 1$ if $x \in A$ and $= 0$ if $x \notin A$. Then, f_n converges to 0 on all $x \in [0,1]$, but $\int_{[0,1]} f_n = 1$ for all n. Thus,

$$\lim_n \int_{[0,1]} f_n \, dx = 1 \neq 0 = \int_{[0,1]} \lim_n f_n. \qquad \square$$

Remark. There cannot be such an example if $\{f_n\}$ is uniformly bounded.

Problem 2. Provide a counter-example that shows a map $f : X \to Y$ and two sets $A, B \subseteq X$ such that

$$f(A \cap B) \neq f(A) \cap f(B).$$

Solution. Let $X = Y = \{0,1\}$ and consider $f : \{0,1\} \to \{0,1\}$ that $f(0) = f(1) = 1$. Then, for $A = \{0\}$ and $B = \{1\}$, we get $f(A \cap B) = f(\emptyset) = \emptyset$ but $f(A) \cap f(B) = \{1\}$. $\square$

Problem 3. Determine whether the following collection of subsets on $\mathbb{R}$ defines a σ-algebra on it, and prove your statements:

(a) $\mathcal{F}_1 = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a] : a \in \mathbb{R}\}$.

(b) $\mathcal{F}_2 = \{\emptyset, \mathbb{R}, (0,1) \cup (2,3)\}$.

(c) $\mathcal{F}_3 = \{A, \mathbb{R} \setminus A : A \subset \mathbb{R}, |A| = \aleph_0\}$.

Solution. (a) No. $\mathbb{R} \setminus (-\infty, a] = (a, \infty) \notin \mathcal{F}_1$.

(b) No. $\mathbb{R} \setminus ((0,1) \cup (2,3)) \notin \mathcal{F}_2$.

(c) No. $(\mathbb{R} \setminus \{0,1,2,\cdots\}) \cup ($~~$\mathbb{R} \setminus$~~ $\{1,2,\cdots\}) = \mathbb{R} \setminus \{0\} \notin \mathcal{F}_3$; thanks to *Lydia Boubendir*. $\square$

Remark. $\mathcal{F}_1$ generates Borel σ-algebra on $\mathbb{R}$, $\mathcal{F}_2 = \{\emptyset, \mathbb{R}, A\}$ generates $\{\emptyset, \mathbb{R}, A, \mathbb{R} \setminus A\}$, and $\mathcal{F}_3$ generates $\{A, \mathbb{R} \setminus A : A \subset \mathbb{R}, |A| \leq \aleph_0\}$.

Problem 4. Prove that the arbitrary intersection of σ-algebras is again a σ-algebra.

Solution. Suppose X is a non-empty set and $\{\mathfrak{M}_i\}_{i \in I}$ is a collection of σ-algebras on X. Let $\mathfrak{M} = \cap_{i \in I} \mathfrak{M}_i$. We verify definition of $\mathfrak{M}$ being σ-algebra as follows:

($X \in \mathfrak{M}$) $X \in \mathfrak{M}_i$ for all $i \in I$, so $X \in \mathfrak{M}$.

(closed under complement) $A \in \mathfrak{M} \implies A \in \mathfrak{M}_i$ for all $i \in I \implies X \setminus A \in \mathfrak{M}_i$ for all $i \in I \implies X \setminus A \in \mathfrak{M}$.

(closed under countable union) $\{A_n\}_{n \in \mathbb{N}} \subset \mathfrak{M} \implies \{A_n\}_{n \in \mathbb{N}} \subset \mathfrak{M}_i$ for all $i \in I \implies \cup_{n \in \mathbb{N}} \in \mathfrak{M}_i$ for all $i \in I \implies \cup_{n \in \mathbb{N}} \in \mathfrak{M}$. $\square$

Problem 5. Find an example of a (finite or infinite) collection of σ-algebras such that their union is not a σ-algebra.

Solution. Let $X = \{1,2,3\}$, $\mathfrak{M}_1 = \{\emptyset, \{1\}, \{2,3\}, X\}$, and $\mathfrak{M}_2 = \{\emptyset, \{2\}, \{1,3\}, X\}$. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are σ-algebras on X, but $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$ is not a σ-algebra since $\{1\}, \{2\} \in \mathfrak{M}$ but $\{1\} \cup \{2\} \notin \mathfrak{M}$. $\square$

Problem 6. Prove that the collection of open balls,

$$B_\epsilon(x) \equiv \{y \in \mathbb{R}^n : \|x - y\| < \epsilon\}$$

is a basis for the standard topology on $\mathbb{R}^n$, where ϵ ranges over $(0, \infty)$ and x ranges over $\mathbb{R}^n$. To do that, please state the definition of a basis for a topology.

Solution. Definition of a basis for a topology: Suppose X is a set and $\mathscr{T}$ is a topology on X. We say a family $\mathcal{B} \subseteq \mathscr{T}$ is a *basis* for the topology $\mathscr{T}$ if every open set can be represented as the union of elements in $\mathcal{B}$.

Proof that balls form a basis for the standard topology on $\mathbb{R}^n$: Suppose an open set $U \subseteq \mathbb{R}^n$ is given. For each $x \in U$, there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subseteq U$. So,

$$U = \bigcup_{x \in X} B_{\epsilon_x}(x).$$

Therefore, every open set in $\mathbb{R}^n$ can be represented as the union of open balls. $\square$

Problem 7. Define the extended real line, initially as a set, via

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}.$$

Define a topology on it by providing a basis for its topology using the collection $B_\epsilon(x)$ from above (as ϵ ranges over $(0, \infty)$ and x ranges over $\mathbb{R}$) together with two additional collections

$$\{(a, \infty] : a \in \mathbb{R}\}$$

and

$$\{[-\infty, a) : a \in \mathbb{R}\}.$$

Show that every open set in $\overline{\mathbb{R}}$ thus defined is a countable union of these basis elements.

Solution. Fix an open set $U \subsetneq \overline{\mathbb{R}}$. Denote the collection of basis elements by $\mathfrak{B}$ and let $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\pm\infty\}$. For each $x \in U \cap \overline{\mathbb{Q}}$, there exists the *maximal* basis element A_x that contains x and $A_x \subseteq U$. Explicitly, define

$$A_x = \bigcup_{A \in \mathfrak{B},\, x \in A,\, A \subseteq U} A.$$

Since A_x is the union of intervals that contain x, it is again an interval that contains x. The endpoints are open or $\pm\infty$. Thus, $A_x \in \mathfrak{B}$ and $A_x \subseteq U$ by definition. Let

$$V = \cup_{r \in U \cap \overline{\mathbb{Q}}} A_r.$$

It suffices to show that $U = V$. Since $U \supseteq A_r$ for each r, we get $U \supseteq V$. Now, it is enough to prove that $x \in U$ implies $x \in V$.

For $x \in U \cap \overline{\mathbb{Q}}$, we know $x \in A_x$, so $x \in V$.
For $x \in U \setminus \overline{\mathbb{Q}}$, there exists an open interval $B \subseteq U$ that contains x. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, B contains a rational r. Then, $B \subseteq A_r$ by maximality of A_r. Therefore, $x \in A_r$ for some r and thus $x \in V$. □

Problem 8. Does there exist an infinite σ-algebra which has only countably many elements?

Solution. No. For the sake of contradiction, suppose there is a σ-algebra $\mathfrak{M}$ on a set X such that $\mathfrak{M}$ is countable.

For each $x \in X$, let F_x be the intersection of every element in $\mathfrak{M}$ that contains x. Since $\mathfrak{M}$ was countable, there are at most countably many sets that contain ~~$\mathfrak{M}$~~x (Thanks to Olivia Kwon). Hence, $F_x \in \mathfrak{M}$ because a σ-algebra is closed under a countable intersection.

Claim 1. $y \in F_x \implies F_x = F_y$.
Proof of Claim. First, $F_y \subseteq F_x$ by definition of F_y. If $x \notin F_y$, then $x \in F_x \setminus F_y$ but $F_x \setminus F_y$ is a proper subset of F_x. It contradicts the definition of F_x. Therefore, $x \in F_y$ and thus $F_x \subseteq F_y$. Summing up with the first line, we get $F_x = F_y$. □

Let $\mathcal{F} = \{F_x : x \in X\}$. Then, $\mathcal{F}$ is a subcollection of $\mathfrak{M}$. Due to Claim 1, two distinct elements in $\mathcal{F}$ are necessarily disjoint.

Claim 2. $A = \cup_{x \in A} F_x$.
Proof of Claim. Since $x \in F_x$ for each $x \in A$, we get $A \subseteq \cup_{x \in A} F_x$. On the other hand, from the definition of F_x, we know $F_x \subseteq A$ for all $x \in A$. Thus, $A \supseteq \cup_{x \in A} F_x$. □

Since $\mathfrak{M}$ is countable, $\mathcal{F}$ is at most countable. From Claim 2, every element in $\mathfrak{M}$ is a union of elements in $\mathcal{F}$. Since $\mathcal{F}$ is at most countable, the union of any subcollection of elements in $\mathcal{F}$ is an element of $\mathfrak{M}$. Therefore, $|\mathfrak{M}| = |\mathscr{P}(\mathcal{F})|$. Since the cardinality of a power set cannot be countable, it deduces a contradiction. □

Remark (Proof that $|\mathscr{P}(\mathbb{N})|$ is uncountable). Suppose not. Then $\mathscr{P}(\mathbb{N}) = \{A_k : k \in \mathbb{N}\}$. Define

$$B = \{k \in \mathbb{N} : k \notin A_k\}.$$

Since $B \in \mathscr{P}(\mathbb{N})$, it should be $B = A_n$ for some $n \in \mathbb{N}$. If $n \in B$, then $n \notin A_n$ and if $n \notin B$, then $n \in A_n$. In either case, we get a contradiction.

Problem 9. Show that if $f : X \to \mathbb{R}$ with X a measurable space, and

$$f^{-1}([r, \infty)) \in (X) \qquad (r \in \mathbb{Q})$$

then f is in fact measurable.

Solution. Consider $\mathfrak{F}$ be a family of subsets in $\mathbb{R}$ defined as

$$\mathfrak{F} = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \text{Measurable}(X)\}.$$

Then, $\mathfrak{F}$ is a σ-algebra on $\mathbb{R}$. First, since X is measurable in X, we get $\mathbb{R} \in \mathfrak{F}$. Next, if $A \in \mathfrak{F}$, then $f^{-1}(\mathbb{R} \setminus A) = X \setminus f^{-1}(A)$ is measurable in X, so $\mathbb{R} \setminus A \in \mathfrak{F}$. Lastly, if $\{A_n\} \subset \mathfrak{F}$, then $f^{-1}(\cup_n A_n) = \cup_n f^{-1}(A_n)$ is measurable in X, so $\cup_n A_n \in \mathfrak{F}$.

From the given condition, we know that $[r, \infty) \in \mathfrak{F}$ for all $r \in \mathbb{Q}$. Since $\{[r, \infty) : r \in \mathbb{Q}\}$ generates the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, the σ-algebra $\mathfrak{F}$ must contain every Borel set. Therefore, $f^{-1}(A)$ is measurable in X for every $A \in \mathcal{B}(\mathbb{R})$ which means $f : X \to \mathbb{R}$ is measurable. □

Remark (Proof that $\{[r, \infty) : r \in \mathbb{Q}\}$ generates $\mathcal{B}(\mathbb{R})$). Let $\mathfrak{M}$ be the smallest σ-algebra that contains $[r, \infty)$ for all $r \in \mathbb{Q}$.
i. $(a, \infty) = \cup_{r>a, r\in\mathbb{Q}}[r, \infty) \in \mathfrak{M}$ for any $a \in \mathbb{R}$.
ii. $(a, b] = (a, \infty) \setminus (b, \infty) \in \mathfrak{M}$ for any $a, b \in \mathbb{R}$.
iii. $(a, b) = \cup_{n\in\mathbb{N}}(a, b - n^{-1}] \in \mathfrak{M}$ for any $a, b \in \mathbb{R}$.
iv. Since any open set in $\mathbb{R}$ is a countable union of open intervals, every open set is an element of $\mathfrak{M}$.
v. Therefore, $\mathfrak{M} = \sigma(\text{Open}(X)) = \mathcal{B}(\mathbb{R})$.

Problem 10. Let $f, g : X \to \overline{\mathbb{R}}$ be given measurable functions with X a measurable space. Show that

$$\{x \in X : f(x) < g(x)\}, \{x \in X : f(x) = g(x)\}$$

are measurable.

Solution. If $f(x) < g(x)$, then there is a rational number between $f(x)$ and $g(x)$. Thus,

$$\{x \in X : f(x) < g(x)\} = \bigcup_{q\in\mathbb{Q}} \{x \in X : f(x) < q < g(x)\} = \bigcup_{q\in\mathbb{Q}} f^{-1}([-\infty, q)) \cap g^{-1}((q, \infty]).$$

Since f and g are measurable functions, $f^{-1}([-\infty, q))$ and $g^{-1}((q, \infty])$ are measurable sets. Hence, $f^{-1}([-\infty, q)) \cap g^{-1}((q, \infty])$ is a measurable set of each $q \in \mathbb{Q}$. Therefore, $\{x \in X : f(x) < g(x)\}$ is a measurable set as a countable union of measurable sets. And we know that the second set is also measurable because

$$\{x \in X : f(x) = g(x)\} = X \setminus (\{x \in X : f(x) < g(x)\} \cup \{x \in X : f(x) > g(x)\}).$$ □