Princeton University
Spring 2025 MAT425: Measure Theory
HW1
Jan 31st 2025

February 2, 2025

1. Provide an example of a sequence of Riemann integrable functions $f_n : [a,b] \to \mathbb{R}$ which converges pointwise but not uniformly to some function $\lim_n f_n : [a,b] \to \mathbb{R}$ which is *also* Riemann integrable, but such that

$$\lim_n \int_{[a,b]} f_n \neq \int_{[a,b]} \lim_n f_n$$

and even both quantities are finite. Conclude that

$$\lim_n \int_{[a,b]} \neq \int_{[a,b]} \lim_n$$

is violated *not only* because of lack of Riemann integrability.

2. Provide a counter example that shows a map $f : X \to Y$ and two sets $A, B \subseteq X$ such that

$$f(A \cap B) \neq f(A) \cap f(B) \ .$$

3. Determine whether the following collection of subsets or $\mathbb{R}$ define a σ-algebra on it, and prove your statements:
   (a) $\mathscr{F}_1 := \{ \varnothing, \mathbb{R} \} \cup \{ (-\infty, a] \mid a \in \mathbb{R} \}$.
   (b) $\mathscr{F}_2 := \{ \varnothing, \mathbb{R}, (0,1) \cup (2,3) \}$.
   (c) $\mathscr{F}_3 := \{ A, \mathbb{R} \setminus A \mid A \subseteq \mathbb{R} : |A| = \aleph_0 \}$.

4. Prove that the arbitrary intersection of σ-algebras is again a σ-algebra.

5. Find an example of a (finite or infinite) collection of σ-algebras such that their union is *not* a σ-algebra.

6. Prove that the collection of open balls,

$$B_\varepsilon(x) \equiv \{ y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon \}$$

is a *basis for the standard topology on* $\mathbb{R}^n$, where ε ranges over $(0, \infty)$ and x ranges over $\mathbb{R}^n$. To do that, please state the definition of a basis for a topology.

7. Define the extended real line, initially as a set, via

$$\overline{\mathbb{R}} \quad := \quad \mathbb{R} \cup \{ \pm\infty \} \ .$$

Define a topology on it *via providing a basis for its topology* using the collection $B_\varepsilon(x)$ from above (as ε ranges over $(0, \infty)$ and x ranges over $\mathbb{R}$) together with two additional collections

$$\{ (a, \infty] \mid a \in \mathbb{R} \}$$

and

$$\{ [-\infty, a) \mid a \in \mathbb{R} \} \ .$$

Show that every open set in $\overline{\mathbb{R}}$ thus defined is a countable union of these basis elements.

8. Does there exist an infinite σ-algebra which has only countably many elements?
9. Show that if $f : X \to \mathbb{R}$ with X a measurable space, and

$$f^{-1}([r, \infty)) \in (X) \qquad (r \in \mathbb{Q})$$

   then f is in fact measurable.
10. Let $f, g : X \to \overline{\mathbb{R}}$ be given measurable functions with X a measurable space. Show that

$$\{ x \in X \mid f(x) < g(x) \}, \{ x \in X \mid f(x) = g(x) \}$$

   are measurable.