Princeton University
Spring 2025 MAT425: Measure Theory
HW10
Apr 25th 2025

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1. Find an example of two probability measures $\mu, \nu : \mathfrak{B}\left(\mathbb{R}^2\right) \to [0,1]$ such that
$$\mu\left(B_1 \times B_2\right) = \nu\left(B_2 \times B_1\right) \qquad \left(B_1, B_2 \in \mathfrak{B}\left(\mathbb{R}^2\right)\right)$$
fails.

2. Find an example of two probability measures $\mu : \mathfrak{B}\left(\mathbb{R}^2\right) \to [0,1]$ and $\nu : \mathfrak{B}(\mathbb{R}) \to [0,1]$ such that
$$\mu\left(B \times \mathbb{R}\right) = \nu(B) \qquad \left(B \in \mathfrak{B}(\mathbb{R})\right)$$
fails.

3. Prove that the Kolmogorov consistency conditions hold for the SRW (recall the SRW $S_N := \sum_{n=1}^{N} X_n$ is defined indirectly by first defining its sequence of increments $\{ X_n \}_n$) where the marginals are specified via
$$\mathbb{P}_{\left(X_{j_1}, \cdots, X_{j_n}\right)} := \prod_{k=1}^{n} \mu_0 \qquad \left(j_1, \cdots, j_n \in \mathbb{N}\right)$$
for some a-priori probability measure $\mu_0 : \mathfrak{B}(\mathbb{R}) \to [0,1]$.

4. Let $m : [0,T] \to \mathbb{R}$ be a given function and C some linear operator $L^2([0,T] \to \mathbb{R})$ which has an integral kernel $\left(C(t,s)\right)_{(t,s)}$. As an operator, we consider C to be self-adjoint and positive ($C \geq 0$). A Gaussian stochastic process $\left(B_t\right)_{t \in [0,T]}$ associated to m and C is specified by specifying the finite marginals; for any $0 \leq t_1 < \cdots < t_n$,
$$\frac{\mathrm{d}\mathbb{P}_{\left(B_{t_1}, \cdots, B_{t_n}\right)}}{\mathrm{d}\lambda}(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det(K)}} \exp\left(-\frac{1}{2}\left\langle (x-\mu), K^{-1}(x-\mu)\right\rangle_{\mathbb{R}^n}\right) \qquad \left(x \in \mathbb{R}^n\right)$$
where $K_{ij} := C\left(t_i, t_j\right)$ and $\mu_i := m\left(t_i\right)$.
   (a) Show that any Gaussian process obeys the Kolmogorov consistency conditions.
   (b) Show that a Gaussian random vector X in $\mathbb{R}^n$ is uniquely determined by its first two moments $\mathbb{E}\left[X_i\right]$ and $\mathbb{E}\left[X_i X_j\right]$. Conclude that $\left(B_t\right)_{t \in [0,T]}$ is uniquely characterized by saying it is Gaussian and specifying $t \mapsto \mathbb{E}\left[B_t\right]$ and $(s,t) \mapsto \mathbb{E}\left[B_t B_s\right]$. What are these two moments in terms of m and C?

5. What is the operator
$$C : L^2([0,T] \to \mathbb{R}) \to L^2([0,T] \to \mathbb{R})$$
so that
$$C(t,s) = \min(\{ t, s \})\,?$$
Conclude a more "appealing" expression for
$$\frac{\mathrm{d}\mathbb{P}_{\left(B_{t_1}, \cdots, B_{t_n}\right)}}{\mathrm{d}\lambda}(x)$$
when $\left(B_t\right)_t$ is Brownian motion then.

6. Let $\{ X_n : \Omega \to \mathbb{R} \}_{n\in\mathbb{N}}$ be an IID sequence of random variables. Let $A_N := \frac{1}{N}\sum_{n=1}^{N} X_n$.

   (a) Prove that $\{ \mathbb{P}_{A_N} \}_{N\in\mathbb{N}}$ obeys an LDP (with which rate function?).

   (b) Calculate the rate function explicitly for the case that each X_n is a symmetric coin flip for ± 1.

   (c) Use this to calculate the asymptotics of the tail bound

   $$\mathbb{P}\left[\sum_{n=1}^{N} X_n \geq Na\right]$$

   for large N.

   (d) Let $f : \mathbb{R} \to \mathbb{R}$ continuous and bounded, calculate asymptotics of

   $$\mathbb{E}_{A_N}\left[\exp\left(Nf\left(A_N\right)\right)\right] .$$

7. Prove the scaling property of Brownian motion, i.e., that

$$B_t \overset{d}{=} \sqrt{c} B_{\frac{t}{c}} \qquad (c > 0) .$$

8. Assume that scaled Brownian motion obeys an LDP. That is, let $(B_t)_{t\in[0,1]}$ be standard Brownian motion (which is conditioned so that $B_0 = 0$). For any $\varepsilon > 0$, consider $B_t^{(\varepsilon)} := \sqrt{\varepsilon} B_t$, a new stochastic process. The stochastic process $\left(B_t^{(\varepsilon)}\right)_{t\in[0,1]}$ obeys an LDP, in the following sense: Let $\Omega := C_0([0,1])$ be the space of continuous functions which are zero at zero (assume that B takes values in Ω almost-surely). Define $I : \Omega \to [0,\infty]$ via

$$I(\varphi) := \begin{cases} \frac{1}{2}\int_0^1 \varphi'(t)^2 \, \mathrm{d}\lambda(t) & \varphi \text{ is abs. cont. with } \varphi(0) = 0 \text{ and } \varphi' \in L^2 \\ \infty & \text{else} \end{cases} .$$

Then we may consider the measure $\mathbb{P}_B$ as a measure on Ω, and in that sense, $\{ \mathbb{P}_{B^{(\varepsilon)}} \}_{\varepsilon>0}$ obeys an LDP with the above rate function.

Use this assumption and Varadhan's lemma to calculate the asymptotics, as $\lambda \to \infty$, of

$$I_\lambda(x,y) := \mathbb{E}\left[\exp\left(-\lambda^2 \int_0^{\frac{T}{\lambda}} V\left(x + \lambda(y-x)\frac{t}{T} + B_t - \lambda\frac{t}{T}B_T\right)\mathrm{d}t\right)\right]$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion with $B_0 = 0$. Here $V : \mathbb{R} \to \mathbb{R}$ is some function (potential) so that the whole integral is a continuous and bounded function of Ω.

   (a) Step 1: Compare Brownian motion with *pinned Brownian motion*

   $$B_t^{pinned} := x + (y-x)\frac{t}{T} + B_t - \frac{t}{T}B_T$$

   which has the conditioning that $B_0 = x$ *and* $B_T = y$.

   (b) Step 2: Show that $\left(B_t^{\text{pinned}}\right)_t$ also obeys an LDP.

   (c) Step 3: Rewrite $I_\lambda(x,y)$ in terms of the expectation on B_t^{pinned}.

9. Let $(B_t)_{t\geq 0}$ be standard Brownian motion with $B_0 = 0$ and $a > 0$. Define

$$\tau_a := \inf\left(\{ t \geq 0 \mid B_t = a \}|\right) .$$

Define the process

$$\tilde{B}_t := \begin{cases} B_t & t \leq \tau_a \\ 2a - B_t & t > \tau_a \end{cases} .$$

   (a) Show that $\mathbb{P}[\tau_a < \infty] = 1$. In fact calculate $\mathbb{P}[\tau_a \leq t]$.

   (b) Show that $\tilde{B}_t \overset{d}{=} B_t$.

(c) Use this to calculate the tail bound that Brownian motion escapes a box:

$$\mathbb{P}\left[\max_{s\in[0,t]} B_s \geq a\right].$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded, and set

$$u(t,x) := \mathbb{E}\left[f(x+B_t)\right].$$

Show that u satisfies the heat equation

$$\partial_t u = \frac{1}{2}\partial_x^2 u$$

with initial data $u(0,\cdot) = f$.