Princeton University

Spring 2025 MAT425: Measure Theory

HW2 Sample Solutions

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Problem 1. Let X be countable. Show that if X is considered as a measurable space with the σ-algebra $\mathcal{P}(X)$, and $c : \mathcal{P}(X) \to [0, \infty]$ is the counting measure on it, then

$$\int_A f dc = \sum_{x \in A} f(x)$$

for any $A \subseteq X$ and $f : X \to \mathbb{C}$ measurable.

Solution. See Example 2.32 of the lecture notes for the solution. □

Problem 2. Let X be a measurable space and $x_0 \in X$. Show that if $\delta_{x_0} : \text{Msrbl}(X) \to [0, \infty]$ is the Dirac delta (unit mass) measure then

$$\int_A f d\delta_{x_0} = \chi_A(x_0) f(x_0)$$

for any $A \in \text{Msrbl}(X)$ and $f : X \to \mathbb{C}$ measurable.

Solution. See Example 2.33 of the lecture notes for the solution. □

Problem 3. Show that the inequality in Fatou's lemma may well be strict with the following sequence of functions

$$f_n = \begin{cases} \chi_E & n \in 2\mathbb{N} + 1 \\ \chi_{E^c} & n \in 2\mathbb{N} \end{cases}$$

Solution. According to Fatou's lemma we have that:

$$\int_X \big(\liminf_n f_n\big) d\mu \leq \liminf_n \int_X f_n d\mu$$

For the above sequence of functions we have that $\liminf_n f_n(x) = 0$ for all $x \in X$, so the left hand side in the above inequality is identically zero. On the other hand we see that:

$$\int_X f_n d\mu = \begin{cases} \mu(E) & n \in 2\mathbb{N}+1 \\ \mu(E^c) & n \in 2\mathbb{N} \end{cases}$$

Thus, we have that $\liminf_n \int_X f_n d\mu = \min\big(\mu(E), \mu(E^c)\big)$, so the inequality in Fatou's lemma is strict. □

Problem 4. *(Continuity of the integral)* For any $f \in L^1(\mu)$ and $\epsilon > 0$ there exists some $\delta > 0$ such that if $E \in \mathrm{Msrbl}(X)$ is such that $\mu(E) < \delta$ then $\int_E |f| d\mu < \epsilon$.

Solution. (see Proposition 1.12 in Chapter 2 of Stein and Shakarchi: Real analysis) By replacing f with $|f|$ we may assume without loss of generality that $f \geq 0$. We define:

$$E_n = \big\{x : \ f(x) \leq n\big\}, \ f_n(x) = f(x)\chi_{E_n}$$

We have that $f_n \geq 0$ is measurable and $f_n(x) \leq f_{n+1}(x)$. By the monotone convergence theorem we have:

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

For any $\epsilon > 0$, there exists $n > 0$ large enough such that:

$$\int |f - f_n| d\mu < \frac{\epsilon}{2}$$

We also consider $\delta > 0$ small enough such that $2n\delta < \epsilon$. We conclude by:

$$\int_E f d\mu \leq \int |f - f_n| d\mu + \int_E |f_n| d\mu < \frac{\epsilon}{2} + n\delta < \epsilon.$$

□

Problem 5. State and prove the reverse Fatou's lemma (involving lim sup instead of lim inf). What is the additional condition that one must assume compared to the original Fatou?

Solution. We first state the reverse Fatou's lemma: Let $\{f_n\}$ be a sequence of non-negative measurable functions on X. If there exists a non-negative integrable function such that $f_n \leq g$ for all n, then:

$$\limsup_n \int_X f_n d\mu \leq \int_X \big(\limsup_n f_n\big) d\mu$$

To prove this, we consider the sequence of non-negative functions $g_n = g - f_n$ (here we used the additional condition). According to Fatou's lemma we have that:

$$\int_X \big(\liminf_n g_n\big) d\mu \leq \liminf_n \int_X g_n d\mu$$

Also, we see that $\liminf_n g_n = g - \limsup_n f_n \geq 0$ and $\int_X \big(\liminf_n g_n\big) d\mu = \int_X g d\mu - \int_X \big(\limsup_n f_n\big) d\mu$. Moreover, $\liminf_n \int_X g_n d\mu = \int_X g d\mu - \limsup_n \int_X f_n d\mu$, which allows us to conclude.

The requirement that the sequence $\{f_n\}$ is bounded by an integrable function g is necessary. For example, we can take $f_n(x) = n\chi_{(0,1/n)}(x)$, which has $\limsup_n \int f_n dx = 1$. However, $\limsup_n f_n$ is identically zero, so the reverse Fatou's lemma does not apply for this sequence. □

Problem 6. *(Cartesian product of measure spaces)* Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of non-empty sets and set $X = \prod_{\alpha \in A} X_\alpha$. Let $\pi_\alpha : X \to X_\alpha$ be the canonical projections. If we furnish each X_α with the σ-algebra $\mathrm{Msrbl}(X_\alpha)$ then:

$$\big\{\pi_\alpha^{-1}(E_\alpha)|\ E_\alpha \in \mathrm{Msrbl}(X_\alpha),\ \alpha \in A\big\}$$

generates the σ-algebra $\mathrm{Msrbl}(X)$ on X. Show that if A is countable then this σ-algebra equals that generated by

$$\Big\{\prod_{\alpha \in A} E_\alpha|\ E_\alpha \in \mathrm{Msrbl}(X_\alpha)\Big\}$$

Solution. One direction is immediate since we have the inclusion:

$$\big\{\pi_\alpha^{-1}(E_\alpha)|\ E_\alpha \in \mathrm{Msrbl}(X_\alpha),\ \alpha \in A\big\} \subset \Big\{\prod_{\alpha \in A} E_\alpha|\ E_\alpha \in \mathrm{Msrbl}(X_\alpha)\Big\}$$

For the other direction we use the fact that:

$$\prod_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha).$$

□

Problem 7. Show that $\mathcal{B}(\mathbb{R}^n)$ equals the above construction if we consider $\mathbb{R}^n = \prod_{\alpha \in \{1,\ldots,n\}} \mathbb{R}$ and on each copy of $\mathbb{R}$ we choose the σ-algebra $\mathcal{B}(\mathbb{R})$.

Solution. From the previous exercise we know that the σ-algebra $\mathrm{Msrbl}\big(\prod_{\alpha \in \{1,\ldots,n\}} \mathbb{R}\big)$ is generated by:

$$\Big\{\prod_{\alpha \in \{1,\ldots,n\}} E_\alpha|\ E_\alpha \in \mathcal{B}(\mathbb{R})\Big\}$$

The key point is to show that both σ-algebras are generated by:

$$\Big\{\prod_{\alpha \in \{1,\ldots,n\}} I_\alpha|\ I_\alpha \text{ open interval of } \mathbb{R}\Big\}$$

In order to prove that $\mathrm{Msrbl}\big(\prod_{\alpha \in \{1,\ldots,n\}} \mathbb{R}\big)$ is generated by the cartesian product of open intervals we use the familiar fact that any open subset of $\mathbb{R}$ can be written (uniquely) as a countable union of disjoint open intervals. This is the content of Theorem 1.3 in Chapter 1 of Stein and Shakarchi: Real analysis. Next, to show that $\mathcal{B}(\mathbb{R}^n)$ is generated by the cartesian product of open intervals, we use the fact that every open subset of $\mathbb{R}^n$ can be written as a countable union of almost disjoint cubes. This is the content of Theorem 1.4 in Chapter 1 of Stein and Shakarchi: Real analysis. □

Problem 8. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let

$$\mathfrak{N} := \big\{N \in \mathfrak{M} |\ \mu(N) = 0\big\}$$

and

$$\overline{\mathfrak{M}} := \big\{E \cup F |\ E \in \mathfrak{M} \text{ and } \exists N \in \mathfrak{N} \text{ such that } F \subseteq N\big\}$$

Then $\overline{\mathfrak{M}}$ is a σ-algebra in X and $\exists!$ measure $\overline{\mu}$ which extends μ to $\overline{\mathfrak{M}}$. It is called the completion of μ.

Solution. (See Theorem 1.9 in Chapter 1 of Folland: Real Analysis) Since $\mathfrak{M}$ and $\mathfrak{N}$ are closed under countable unions, so is $\overline{\mathfrak{M}}$. If $E \cup F \in \overline{\mathfrak{M}}$ where $E \in \mathfrak{M}$ and $F \subseteq N \in \mathfrak{N}$, we can assume that $E \cap N =$ (otherwise, replace F and N by $F \backslash E$ and $N \backslash E$). Then $E \cup F = (E \cup N) \cap (N^c \cup F)$, so $(E \cup F)^c = (E \cup N)^c \cup (N \backslash F)$. But $(E \cup N)^c \in \mathfrak{M}$ and $N \backslash F \subset N$, so that $(E \cup F)^c \in \overline{\mathfrak{M}}$. Thus, $\overline{\mathfrak{M}}$ is a σ-algebra.

If $E \cup F \in \overline{\mathfrak{M}}$ as above, we define $\overline{\mu}(E \cup F) = \mu(E)$. This is well defined, since if $E_1 \cup F_1 = E_2 \cup F_2$, then $E_1 \subset E_2 \cup N_2$, so $\mu(E_1) \leq \mu(E_2)$. Similarly, we also get $\mu(E_2) \leq \mu(E_1)$, showing that $\overline{\mu}$ is well defined. Showing countable additivity is immediate. Finally, we consider any other extension $\tilde{\mu}$ of μ to $\overline{\mathfrak{M}}$, so $\tilde{\mu}(E) = \mu(E)$ for any $E \in \mathfrak{M}$. In particular, for any $N \in \mathfrak{N}$ we have $\tilde{\mu}(N) = 0$. Thus, for any $F \subset N \in \mathfrak{N}$ we have $\tilde{\mu}(F) = 0$ and we conclude that $\tilde{\mu} = \overline{\mu}$. $\square$

Problem 9. Show that if $a_1, \ldots, a_n \in [0, \infty)$ and $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathfrak{M})$ then $\sum_{j=1}^{n} a_j \mu_j$ is a measure on $(X, \mathfrak{M})$ too.

Solution. We check that the new map is countable additive. For any pairwise disjoint measurable sets $A_1, A_2, \ldots$ we have:

$$\sum_{j=1}^{n} a_j \mu_j \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{j=1}^{n} a_j \left(\sum_{i=1}^{\infty} \mu_j(A_i) \right) = \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_j \mu_j(A_i)$$

$\square$