Princeton University
Spring 2025 MAT425: Measure Theory
HW2
Feb 9th 2025

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1. Let X be countable. Show that if X is considered as a measurable space with the σ-algebra $\mathscr{P}(X)$, and $c : \mathscr{P}(X) \to [0, \infty]$ is the counting measure on it, then
$$\int_A f \mathrm{d}c = \sum_{x \in A} f(x)$$
for any $A \subseteq X$ and $f : X \to \mathbb{C}$ measurable.

2. Let X be a measurable space and $x_0 \in X$. Show that if $\delta_{x_0} : \operatorname{Msrbl}(X) \to [0, \infty]$ is the Dirac delta (unit mass) measure then
$$\int_A f \mathrm{d}\delta_{x_0} = \chi_A(x_0) f(x_0)$$
for any $A \in \operatorname{Msrbl}(X)$ and $f : X \to \mathbb{C}$ measurable.

3. Show that the inequality in Fatou's lemma may well be strict with the following sequence of functions
$$f_n := \begin{cases} \chi_E & n \in 2\mathbb{N}+1 \\ \chi_{E^c} & n \in 2\mathbb{N} \end{cases}.$$

4. (*Continuity of the integral*) For any $f \in L^1(\mu)$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that if $E \in \operatorname{Msrbl}(X)$ is such that$\mu(E) < \delta$ then $\int_E |f| \mathrm{d}\mu < \varepsilon$.

5. State and prove the reverse Fatou's lemma (involving $\limsup$ instead of $\liminf$). What is the additional condition that one must assume compared to the original Fatou?

6. (*Cartesian product of measure spaces*) Let $\{ X_\alpha \}_{\alpha \in A}$ be an indexed collection of non-empty sets and set $\prod_{\alpha \in A} X_\alpha$. Let
$$\pi_\alpha : \prod_{\alpha \in A} X_\alpha \quad \to \quad X_\alpha$$
be the canonical projections. If we furnish each X_α with a σ-algebra $\operatorname{Msrbl}(X_\alpha)$ then
$$\left\{ \pi_\alpha^{-1}(E_\alpha) \;\middle|\; E_\alpha \in \operatorname{Msrbl}(X_\alpha), \alpha \in A \right\}$$
generates *the* σ-algebra $\operatorname{Msrbl}\left(\prod_{\alpha \in A} X_\alpha\right)$ on $\prod_{\alpha \in A} X_\alpha$. Show that if A is countable then this σ-algebra equals that generated by
$$\left\{ \prod_{\alpha \in A} E_\alpha \;\middle|\; E_\alpha \in \operatorname{Msrbl}(X_\alpha) \right\}.$$

7. Show that $\mathscr{B}(\mathbb{R}^n)$ equals the above construction if we consider $\mathbb{R}^n = \prod_{\alpha \in \{ 1, \cdots, n \}} \mathbb{R}$ and on each copy of $\mathbb{R}$ we choose the σ-algebra $\mathscr{B}(\mathbb{R})$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let

$$\mathcal{N} := \{ N \in \mathcal{M} \mid \mu(N) = 0 \}$$

and

$$\overline{\mathcal{M}} := \{ E \cup F \mid E \in \mathcal{M} \wedge F \subseteq N \exists N \in \mathcal{N} \} .$$

Then $\overline{\mathcal{M}}$ is a σ-algebra in X and $\exists!$ measure $\overline{\mu}$ which extends μ to $\overline{\mathcal{M}}$. It is called the *completion* of μ.

9. Show that if $a_1, \cdots, a_n \in [0, \infty)$ and $\mu_1, \cdots, \mu_n$ are measures on $(X, \mathcal{M})$ then $\sum_{j=1}^{n} a_j \mu_j$ is a measure on $(X, \mathcal{M})$ too.