Princeton University
Spring 2025 MAT425: Measure Theory
HW9
Apr 19th 2025

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1. Show that if $X_n \to X$ in total variation (see the lecture notes footnote for a definition) then $X_n \to X$ in distribution.
2. Show that if $X_n \to X$ in probability then $X_n \to X$ in distribution.
3. Show that $X_n \to X$ in distribution iff $\mathbb{P}[X_n < t] \to \mathbb{P}[X < t]$ pointwise in $t \in \mathbb{R}$, for all t such that $t \mapsto \mathbb{P}[X < t]$ is continuous.
4. Show that if $\mathbb{E}\left[e^{itX_n}\right] \to \mathbb{E}\left[e^{itX}\right]$ pointwise in t then $X_n \to X$ in distribution.
5. (*Høffding's lemma*) Using Taylor and Jensen, show that if X is a real-valued random variable such that $a \le X \le b$ almost-surely, then
$$\mathbb{E}\left[e^{tX}\right] \le \exp\left(t\mathbb{E}[X] + \frac{t^2(b-a)^2}{8}\right) \qquad (t \in \mathbb{R}) \ .$$
Also show the trivial lower bound from Jensen,
$$\mathbb{E}\left[e^{tX}\right] \ge e^{t\mathbb{E}[X]} \qquad (t \ge 0) \ .$$
6. (*Paley–Zygmund inequality*) Let $X \ge 0$ be an L^2 random variable. Show that then
$$\mathbb{P}[X \ge \theta\mathbb{E}[X]] \ge (1-\theta)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \qquad (\theta \in [0,1]) \ .$$
7. (*Hölder's equality*)
   (a) Let $0 < r < s < 1$ and $Y \ge 0$ be a random variable. Show that
   $$\mathbb{E}[Y^r] = (\mathbb{E}[Y^s])^{\frac{r}{s}} \exp\left(-\int_{q=0}^{s} f_{r,s}(q)\, \mathbb{Var}_q[\log(Y)]\, d\lambda(q)\right)$$
   where
   $$f_{r,s}(q) := \frac{1}{s}\min(\{r,q\})(s - \max(\{r,q\})) \qquad (q \in (0,s))$$
   and for any random variable X,
   $$\mathbb{Var}_q[X] \equiv \mathbb{E}_q\left[(X - \mathbb{E}_q[X])^2\right] \equiv \frac{\mathbb{E}\left[e^{qX}\left(X - \frac{\mathbb{E}\left[Xe^{qX}\right]}{\mathbb{E}\left[e^{qX}\right]}\right)^2\right]}{\mathbb{E}\left[e^{qX}\right]}, \qquad \mathbb{E}_q[\cdot] := \frac{\mathbb{E}\left[\cdot e^{qX}\right]}{\mathbb{E}\left[e^{qX}\right]} \ .$$
   (b) Now let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative random variables such that for any $s \in (0,1)$ there exists some $C_s < \infty$ such that
   $$\sup_{n\in\mathbb{N}} \mathbb{E}[Y_n^s] \le C_s$$

and such that for any $s \in (0,1)$ there exists some $c_s > 0$ with which

$$\inf_{q \in (0,s)} \mathbb{Var}_q \left[\log\left(Y_n\right)\right] \geq c_s n \qquad (n \in \mathbb{N}) \; .$$

Conclude that for any $r \in (0,1)$,

$$\mathbb{E}\left[Y_n^r\right] \leq D_r \exp\left(-d_r n\right) \qquad (n \in \mathbb{N}) \; .$$

Find optimal $D_r < \infty$ and $d_r > 0$.

8. (*The Layer-Cake Representation* revisited (cf. HW4Q6)) Let $X \geq 0$ be a random variable. Show that

$$\mathbb{E}\left[X^s\right] = s \int_{t=0}^{\infty} \mathbb{P}\left[X > t\right] t^{s-1} \mathrm{d}\lambda\left(t\right) \qquad (s > 0) \; .$$

9. Let X be a real-valued random variable such that there are $0 < \alpha < a$, $\varepsilon \in (0,1)$, $\beta \in (0,\infty)$ with which

$$\mathbb{P}\left[|X| < \alpha\right] \leq \beta \sqrt{\mathbb{P}\left[X \geq a\right] \mathbb{P}\left[X \leq -a\right]} + \varepsilon \, .$$

Show that then, the following lower bound holds

$$\mathbb{E}\left[X^2\right] \geq \frac{1-\varepsilon}{1+\frac{1}{2}\beta}\alpha^2 \, .$$

10. Let $A > 0$ be some $n \times n$ matrix with entries in $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ (recall that $A > 0$ means $\langle v, Av\rangle > 0$ for all $v \in \mathbb{C}^n \setminus \{0\}$; with this notation we mean please carry out the calculation for both real and complex cases). Calculate the following integrals:

   (a) The Gaussian normalization factor:

$$Z_A := \int_{x \in \mathbb{F}^n} \mathrm{e}^{-\frac{1}{2}\langle x, Ax\rangle} \mathrm{d}\lambda\left(x\right) .$$

   (b) The unnormalized Gaussian MGF: For some $v \in \mathbb{F}^n$,

$$Z_A \mathbb{E}_A\left[\mathrm{e}^{\langle v, X\rangle}\right] \equiv \int_{x \in \mathbb{F}^n} \mathrm{e}^{-\frac{1}{2}\langle x, Ax\rangle + \langle v, x\rangle} \mathrm{d}\lambda\left(x\right) .$$

   (c) The Gaussian two point function: For some $v_1, v_2 \in \mathbb{F}^n$,

$$\mathbb{E}_A\left[\langle v_1, X\rangle \langle X, v_2\rangle\right] \; .$$

11. Let $\{X_n\}_{n \in \mathbb{N}}$ be an IID sequence of Bernoulli random variables, each with parameter $p \in (0,1)$.

   (a) Calculate the asymptotic distribution of the random variable

$$A_N := \frac{1}{N} \sum_{n=1}^{N} X_n$$

   as $N \to \infty$ by invoking the central limit theorem.

   (b) Repeat this exercise by proving (using Striling) and then invoking the "De Moivre–Laplace theorem":

$$\binom{n}{k} p^k q^{n-k} \cong \frac{1}{\sqrt{2\pi npq}} \mathrm{e}^{-\frac{(k-np)^2}{2npq}} \qquad (n \in \mathbb{N}, p+q=1;\, p,q>0) \; .$$

12. Let Z be a standard normal random variable distributed in $\mathcal{N}(0,1)$ and $\mu \in \mathbb{R}, \sigma > 0$. Define a new random variable

$$X := \mathrm{e}^{\mu + \sigma Z} \, .$$

We say that X is a *log-normal random variable* with distribution parameters μ, σ.

   (a) Calculate $\mathbb{E}\left[X^n\right]$ for all $n \in \mathbb{N}_{\geq 0}$ (there is a simple closed-form formula) and show

$$\mathbb{E}\left[X^n\right] = \mathrm{e}^{n\mu + \frac{1}{2}\sigma^2 n^2} \qquad (n \in \mathbb{N}) \; .$$

(b) Show that

$$\mathbb{E}\left[\mathrm{e}^{tX}\right] \quad = \quad \infty \qquad (t > 0) \ .$$

(c) Define a measure

$$\nu := \sum_{k=1}^{\infty} p_k \delta_{x_k}$$

where $\{ x_k \}_{k \in \mathbb{N}} \subseteq (0, \infty)$ is some sequence and $\{ p_k \}_{k \in \mathbb{N}} \subseteq (0, \infty)$ is chosen so that $\sum_{k=1}^{\infty} p_k = 1$ and

$$\sum_{k=1}^{\infty} p_k x_k^n = \mathbb{E}\left[X^n\right] \qquad (n \in \mathbb{N}) \ .$$

Conclude that ν and $\mathbb{P}_X$ have the same sequence of moments but they are *not* the same measure.