

Functional Analysis
Princeton University MAT520
HW1, assigned on Sep 5 2025

September 8, 2025

Note: I *know* this is a lot. Do as many as doesn't make you suffer. Skip over those exercises that you've done in the past.

1 Topological vector spaces

1. Prove that $\mathbb{C}^n$ with its Euclidean topology is a topological vector space, i.e., show that vector addition and scalar multiplication are continuous with respect to the Euclidean topology.
2. Provide an example for a topological vector space that is not T_1 .
3. Prove that $\mathbb{C}$ with the French metro metric is *not* homeomorphic (=topologically isomorphic) to $\mathbb{C}$ (when considered as topological spaces) with the Euclidean metric. Conclude (why?) that $\mathbb{C}$ with the French metro metric is *not* a TVS.
4. Let $C := \{ f : [0, 1] \rightarrow \mathbb{C} \mid f \text{ is continuous} \}$ and define

$$d(f, g) := \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

Show that d is a metric on C , show that C is a vector space (with pointwise addition and scalar multiplication), and show that the topology which d induces on C makes it into a TVS. Show that this TVS has a countable local base.

5. Let X be the VS of all continuous functions $f : (0, 1) \rightarrow \mathbb{C}$. For any $f \in X$ and $r > 0$, set

$$V(f, r) := \{ g \in X \mid |g(x) - f(x)| < r \forall x \in (0, 1) \}$$

and set $\text{Open}(X)$ as the topology generated by $\{ V(f, r) \}_{f \in X, r > 0}$ (is this collection a basis or a sub-basis for a topology?). Show that w.r.t. $\text{Open}(X)$, vector addition is continuous but scalar multiplication is *not*.

6. Show that $\mathbb{C}^{\mathbb{N}}$ with the product topology is a TVS.
7. Let $X := C((0, 1) \rightarrow \mathbb{C})$ and on it define $\text{Open}(X)$ as the initial topology generated by the family of evaluations $f \mapsto f(t)$ as t ranges in $(0, 1)$. Prove that X is a TVS, but that as a topological space it is *not* metrizable. Define now $\widetilde{\text{Open}}(X)$ to be the initial topology generated by the family of evaluations $f \mapsto f(t)$ as t ranges in $(0, 1) \cap \mathbb{Q}$. Show this latter space is a *metrizable* TVS.
8. Show that the compact open topology is strictly *weaker* than the sup-norm topology on $C_b([0, \infty) \rightarrow \mathbb{C})$, the set of continuous bounded maps.

2 Normed spaces

9. Let a normed vector space $(X, \|\cdot\|)$ be given. Show that there exists some sesquilinear inner product

$$\langle \cdot, \cdot \rangle : X^2 \rightarrow \mathbb{C}$$

which is compatible with the norm, in the sense that

$$\|x\| = \sqrt{\langle x, x \rangle} \quad (x \in X)$$

if and only if the norm satisfies (any one of the equivalent) parallelogram identity:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad (x, y \in X) .$$

Actually that if $\leq$ holds above for all $x, y \in X$ then $=$ also holds.

Explicitly show this fails for two vectors in ℓ^p when $p \neq 2$.

10. Let an inner-product vector space $(X, \langle \cdot, \cdot \rangle)$ be given. Prove the Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\| .$$

11. Prove the triangle inequality for the Euclidean norm on $\mathbb{C}^n$ *without* invoking its inner product structure.

12. Two norms $\|\cdot\|_1, \|\cdot\|_2$ on a normed space X are called *equivalent* iff $\exists a, b \in (0, \infty)$ such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \quad (x \in X) .$$

Show that all norms on $\mathbb{C}^n$ are equivalent. Conclude that all norm topologies on $\mathbb{C}^n$ are homeomorphic.

13. Let X be a Banach space which is Banach w.r.t. two different norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that $\|\cdot\|_1 \leq C\|\cdot\|_2$ for some $C \in (0, \infty)$. Show that there exists a $D \in (0, \infty)$ such that $\|\cdot\|_2 \leq D\|\cdot\|_1$.

14. Show that

$$\sup_{x \in X: \|x\| \leq 1} \|Ax\| = \sup_{x \in X: \|x\| = 1} \|Ax\|$$

where X, Y are Banach spaces and $A : X \rightarrow Y$ is a linear map.

15. Show that a linear map $A : X \rightarrow Y$ between two Banach spaces is bounded iff it is continuous iff it is continuous at zero (in showing continuity implies boundedness please do not use the TVS theorem from chapter 1 but rather do this directly in the context of Banach spaces).

16. Show that if $A : X \rightarrow Y$ is a bounded linear map between Banach spaces then

$$\|Ax\|_Y \leq \|A\|_{\text{op}} \|x\|_X \quad (x \in X) .$$

17. Show that if $A, B : X \rightarrow X$ are bounded linear maps on a Banach space then

$$\|AB\|_{\text{op}} \leq \|A\|_{\text{op}} \|B\|_{\text{op}} .$$

18. Let $L^\infty(\mathbb{R} \rightarrow \mathbb{C})$ be the space of (equivalence classes of) essentially bounded Lebesgue-measurable functions $\mathbb{R} \rightarrow \mathbb{C}$, modulo a.e. equality, with the essential-sup norm. Show it is Banach.

19. Let a normed vector space $(X, \|\cdot\|)$ be given. Show that $(X, \|\cdot\|)$ is complete iff, for *any* sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$,

$$\left(\sum_{n \in \mathbb{N}} \|x_n\| < \infty \right) \implies \left(\lim_{N \rightarrow \infty} \sum_{n=1}^N x_n \text{ } \exists \text{ and equals some } x \in X \right) .$$

20. Show that $\ell^p(\mathbb{N} \rightarrow \mathbb{C})$ is a Banach space for $p \geq 1$. Show that it is *not* TVS isomorphic to any Hilbert space if $p \neq 2$.

21. View $\ell^p(\mathbb{N} \rightarrow \mathbb{C}) \subseteq \mathbb{C}^\mathbb{N}$ with the *product* topology. Show that this topology is strictly *weaker* than the ℓ^p -norm topology.

22. Show that

$$X := \{ a : \mathbb{N} \rightarrow \mathbb{C} \mid |\text{supp}(a)| < \infty \}$$

is dense in $\ell^p(\mathbb{N} \rightarrow \mathbb{C})$ for all $p \in [1, \infty)$ but is *not* dense in $\ell^\infty(\mathbb{N} \rightarrow \mathbb{C})$.

23. Prove $\ell^1 \subsetneq \ell^2 \subsetneq \ell_0 \subsetneq \ell^\infty$ where all inclusions are continuous. Here ℓ_0 is the set of sequences which converge to zero.

24. Show that $C([0, 1] \rightarrow \mathbb{C})$ with $\|\cdot\|_1$ is *not* complete.
25. Show that $C([0, 1] \rightarrow \mathbb{C})$ with $\|\cdot\|_\infty$ is separable, $L^p((0, 1) \rightarrow \mathbb{C})$ is separable for $p \in [1, \infty)$ but $L^\infty((0, 1) \rightarrow \mathbb{C})$ is *not*.
26. On $\ell^p(\mathbb{N} \rightarrow \mathbb{C})$, define R as

$$\begin{aligned}(Ra)_n &:= a_{n-1} & (n \geq 2) \\ (Ra)_1 &:= 0.\end{aligned}$$

Prove that R is bounded with $\|R\| = 1$. In fact show that R is an isometry.

27. On $\ell^1(\mathbb{Z} \rightarrow \mathbb{C})$, and for some fixed $a \in \ell^1(\mathbb{Z} \rightarrow \mathbb{C})$, define C_a via

$$(C_a b)_n := \sum_k a_{n-k} b_k.$$

Show that $\|C_a\|_{\text{op}} \leq \|a\|_1$.

28. Define $V : L^1((0, 1) \rightarrow \mathbb{C}) \rightarrow C([0, 1] \rightarrow \mathbb{C})$ (the latter space to be taken with the supremum norm, i.e., $\|f\|_\infty \equiv \sup_{x \in [0, 1]} |f(x)|$; in this case since $[0, 1]$ this agrees with the compact-open topology) by

$$(Vf)(x) := \int_{t=0}^x f(t) dt.$$

Show Vf is continuous (so V is well-defined). Show that V is a linear bounded operator and that $\|V\|_{\text{op}} \leq 1$.

29. In $L^p((0, 1) \rightarrow \mathbb{C})$ with $p \in [1, \infty)$, let $Y := \left\{ f \in L^p((0, 1) \rightarrow \mathbb{C}) \mid \int_{[0, 1]} f = 0 \right\}$. Show that Y is closed. Then find a natural linear subspace that is dense but not closed.
30. Characterize the relatively compact subsets of $C_b(\mathbb{R} \rightarrow \mathbb{C})$ (bounded continuous functions) with the sup-norm. Test this criterion on translates of a fixed function f .