

MAT520 HW10

December 14, 2025

1. See [Q1](#).
2. See [Q7](#).
3. These follow from the fact that $\mathbb{1} + K$ is Fredholm.
6. See [Q10](#).
7. See [Q11](#).
8. See Example 9.65 in LN.
9. Diagonal Fredholm operator can only have index zero because for diagonal operator D we have $\ker D = (\operatorname{im} D)^\perp$ and the latter is $\ker D^*$.
10. (a) Since $\mathbb{S}^1$ is compact, we have $\|M_f\| \leq \|f\|_\infty$. The map $\mathcal{F}$ is unitary. Thus $\mathcal{F}M_f\mathcal{F}^*$ is bounded.
(b) Let $T_f = \mathcal{F}M_f\mathcal{F}^*$ and $\mathbb{A}T_f = \mathbb{A}T_f\mathbb{A} + \mathbb{A}^\perp$. It suffices to show that $[\mathbb{A}T_f, \mathbb{A}]$ is compact. Indeed, if it is, then $\mathbb{A}T_{1/f}$ is a parametrix for $\mathbb{A}T_f$. If g is a trigonometric polynomial, then one can show that T_g is finite-hopping (all but finitely many diagonal lines are zero), and hence $[T_g, \mathbb{A}]$ is finite-rank. Approximate f continuous uniformly by trigonometric polynomial p_k , then $\|[T_f, \mathbb{A}] - [T_{p_k}, \mathbb{A}]\| \rightarrow 0$ in operator norm. Thus $[T_f, \mathbb{A}]$ is compact.
(c) Let $\mathcal{N}(f) := \operatorname{index}(\mathbb{A}T_f)$ be the Fredholm index of $\mathbb{A}T_f$. Let $\mathcal{S}$ be the space of continuous functions $\mathbb{S}^1 \rightarrow \mathbb{C} \setminus \{0\}$. One can show $\mathcal{N} : \mathcal{S} \rightarrow \mathbb{Z}$ is a group homomorphism. On the other hand, Wind is also a group homomorphism $\mathcal{S} \rightarrow \mathbb{Z}$, see [Theorem 3.18](#). They are both constant on path-connected components of $\mathcal{S}$. Thus there exists a bijection Φ such that $\mathcal{N} = \Phi \circ \operatorname{Wind}$. In fact, Φ is a group homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ and hence take the form $n \mapsto cn$. We determine that $c = -1$ by considering the function $f(z) = z$ in $\mathcal{S}$.

- (d) The polar part of T_f is $U = T_{f/|f|}$. Since $|f| > 0$ it has winding number 0 and hence $\mathcal{N}(f) = 0$. Then

$$\mathcal{N}(f) = \mathcal{N}(f/|f| \cdot |f|) = \mathcal{N}(f/|f|) = \text{index}(\Lambda U)$$

One can show that $[U, \Lambda]$ is trace-class. Then we invoke Lemma 3.7 in the [note](#), to conclude the relation.

11. See [Theorem A.2](#).