

Functional Analysis

Princeton University MAT520

HW11, Assigned on Dec 5 2025

December 6, 2025

1. Let $A = U|A|$ be the polar decomposition of an operator $A \in \mathcal{B}(\mathcal{H})$. Let $f_n : [0, \infty) \rightarrow [0, \infty)$ be given by

$$f_n(x) := \begin{cases} \frac{1}{x} & x \geq \frac{1}{n} \\ n & x \leq \frac{1}{n} \end{cases} \quad (x \geq 0).$$

Prove that

$$U = \text{s-lim}_{n \rightarrow \infty} A f_n(|A|).$$

[extra] Do the same with

$$f_n(x) := \frac{1}{x + \frac{1}{n}} \quad (x \geq 0).$$

2. Prove that if $A \in \mathcal{B}(\mathcal{H})$ is normal then

$$\|A\| = r(A)$$

where $r(A)$ is the spectral radius of an operator.

3. Let $A \in \mathcal{B}(\mathcal{H})$ be normal. Assuming the multiplicative form of the spectral theorem for self-adjoint operators, show there exists some finite measure space (M, μ) and a unitary

$$U : \mathcal{H} \rightarrow L^2(M, \mu)$$

such that there exists a bounded Borel function $f : M \rightarrow \mathbb{C}$ such that

$$(UAU^*\psi)(m) = f(m)\psi(m) \quad (m \in M; \psi \in L^2(M, \mu)).$$

4. Assuming the spectral theorem for self-adjoint operators, show that if $A, B \in \mathcal{B}(\mathcal{H})$ are two self-adjoint operators such that $[A, B] = 0$ then there exists a finite measure space (M, μ) and a unitary

$$U : \mathcal{H} \rightarrow L^2(M, \mu)$$

such that there are two bounded Borel functions $f, g : M \rightarrow \mathbb{R}$ which obey

$$\begin{aligned} (UAU^*\psi)(m) &= f(m)\psi(m) \\ (UBU^*\psi)(m) &= g(m)\psi(m) \end{aligned}$$

for all $m \in M$ and $\psi \in L^2(M, \mu)$.

5. Prove that for $A \in \mathcal{B}(\mathcal{H})$ self-adjoint and $\chi_*(A)$ the projection-valued measure of A , we have

$$\lambda \in \sigma(A) \iff [\chi_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A) \neq 0 \quad (\varepsilon > 0)] \quad (\lambda \in \mathbb{R}).$$

6. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and on it define the discrete Laplacian

$$-\Delta = 2\mathbb{1} - R - R^*$$

where R is the bilateral right shift operator.

(a) Using the Fourier series $\mathcal{F} : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{S}^1)$ given by

$$\psi \mapsto \left([-\pi, \pi] \ni k \mapsto \sum_{x \in \mathbb{Z}} e^{-ikx} \psi =: \widehat{\psi}(k) \right)$$

diagonalize $-\Delta$ to get

$$\mathcal{F}(-\Delta)\mathcal{F}^* = M_{\mathcal{E}}$$

where $\mathcal{E} : \mathbb{S}^1 \rightarrow \mathbb{R}$ is given by $\mathcal{E}(k) = 2 - 2 \cos(k)$ and $M_{\mathcal{E}}$ is the associated multiplication operator. We use the inner product

$$\langle \widehat{\varphi}, \widehat{\psi} \rangle_{L^2(\mathbb{S}^1)} \equiv \frac{1}{2\pi} \int_{k=-\pi}^{\pi} \overline{\widehat{\varphi}(k)} \widehat{\psi}(k) dk.$$

Note that the solution of this equation for $k \in [-\pi, \pi]$ is given by

$$\mathcal{E}^{-1}(E) = \pm \arccos\left(1 - \frac{1}{2}E\right) \quad (E \in [0, 4]). \quad (0.1)$$

Now, if we use

$$\begin{aligned} L^2(\mathbb{S}^1, \text{Lebesgue msr. on circle}) &\cong L^2([-\pi, \pi], \text{Lebesgue msr. } \mathbb{R}) \\ &= L^2_{\text{even}}([-\pi, \pi], \text{Lebesgue msr. } \mathbb{R}) \oplus L^2_{\text{odd}}([-\pi, \pi], \text{Lebesgue msr. } \mathbb{R}) \end{aligned}$$

then, since even and odd functions on $[-\pi, \pi]$ are entirely determined by their values on $[0, \pi]$, within each summand we could solve (0.1). With it, i.e.,

$$\widehat{\varphi} := \widehat{\varphi_{\text{even}}} + \widehat{\varphi_{\text{odd}}}$$

so that

$$\begin{aligned} \langle \psi, f(-\Delta)\varphi \rangle_{\ell^2(\mathbb{Z})} &= \langle \mathcal{F}\psi, \mathcal{F}f(-\Delta)\mathcal{F}^*\mathcal{F}\varphi \rangle_{L^2(\mathbb{S}^1)} \\ &= \langle \widehat{\psi}, f(M_{\mathcal{E}})\widehat{\varphi} \rangle_{L^2(\mathbb{S}^1)} \\ &= \langle \widehat{\psi}, M_{f \circ \mathcal{E}}\widehat{\varphi} \rangle_{L^2(\mathbb{S}^1)} \\ &= \sum_{\alpha=\text{even, odd}} \langle \widehat{\psi}_{\alpha}, M_{f \circ \mathcal{E}}\widehat{\varphi}_{\alpha} \rangle_{L^2_{\alpha}(\mathbb{S}^1)}. \end{aligned}$$

The cross terms vanish since $\mathcal{E}$ is itself even (and thus so is $f \circ \mathcal{E}$). But now

$$\begin{aligned} \langle \widehat{\psi}_{\alpha}, M_{f \circ \mathcal{E}}\widehat{\varphi}_{\alpha} \rangle_{L^2_{\alpha}(\mathbb{S}^1)} &\equiv \frac{1}{2\pi} \int_{k=-\pi}^{\pi} \overline{\widehat{\psi}_{\alpha}(k)} f(\mathcal{E}(k)) \widehat{\varphi}_{\alpha}(k) dk \\ &= \frac{1}{\pi} \int_{k=0}^{\pi} \overline{\widehat{\psi}_{\alpha}(k)} f(\mathcal{E}(k)) \widehat{\varphi}_{\alpha}(k) dk \\ &= \frac{1}{\pi} \int_{E=0}^4 \overline{\widehat{\psi}_{\alpha}(\mathcal{E}^{-1}(E))} f(\mathcal{E}(\mathcal{E}^{-1}(E))) \widehat{\varphi}_{\alpha}(\mathcal{E}^{-1}(E)) d\mathcal{E}^{-1}(E) \\ &= \int_{E=0}^4 f(E) \frac{1}{\pi} \overline{\widehat{\psi}_{\alpha}(\mathcal{E}^{-1}(E))} \widehat{\varphi}_{\alpha}(\mathcal{E}^{-1}(E)) \frac{1}{\sqrt{E(4-E)}} dE. \end{aligned}$$

We thus identify

$$d\mu_{-\Delta, \psi, \varphi}(E) := \frac{1}{\pi} \frac{1}{\sqrt{E(4-E)}} \sum_{\alpha=\text{even, odd}} \overline{\widehat{\psi}_{\alpha}(\mathcal{E}^{-1}(E))} \widehat{\varphi}_{\alpha}(\mathcal{E}^{-1}(E)) dE \quad (0.2)$$

as the spectral measure $\mu_{-\Delta, \psi, \varphi}$ associated to the triplet $-\Delta, \psi, \varphi$.

(b) Recall the definition of a cyclic vector from the lecture notes. For $x \in \mathbb{Z}$, is δ_x a cyclic vector for $-\Delta$?

(c) To make the above more formal, let us define

$$\mathcal{H}_{\text{even}} := \{ \psi \in \ell^2(\mathbb{Z}) \mid \psi_x = \psi_{-x} \} \quad \mathcal{H}_{\text{odd}} := \{ \psi \in \ell^2(\mathbb{Z}) \mid \psi_x = -\psi_{-x} \}.$$

Verify that $-\Delta \mathcal{H}_\alpha \subseteq \mathcal{H}_\alpha$ for $\alpha = \text{even, odd}$. Hence we may concentrate on

$$-\Delta_\alpha \equiv (-\Delta)|_{\mathcal{H}_\alpha} \quad (\alpha = \text{even, odd}).$$

Verify that $\delta_0 \in \mathcal{H}_{\text{even}}$ is a cyclic vector for $-\Delta_{\text{even}}$ and $\delta_1 - \delta_{-1} \in \mathcal{H}_{\text{odd}}$ is a cyclic vector for $-\Delta_{\text{odd}}$. Find now a unitary map

$$U : \ell^2(\mathbb{Z}) \rightarrow L^2([0, 4], \mu) \oplus L^2([0, 4], \mu)$$

such that

$$\left(U(-\Delta) U^* \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right)(E) = \begin{bmatrix} E f_1(E) \\ E f_2(E) \end{bmatrix} \quad (E \in [0, 4], f_1, f_2 \in L^2([0, 4], \mu)).$$

What is $\mu_{-\Delta, \psi, \varphi}$ from before in terms of μ here?

Note: here it is more convenient to decompose the Hilbert space into even and odd since that still looks nice in real space $\ell^2(\mathbb{Z})$. On the other hand if one insists on a decomposition into positive and negative momenta then in real space the decomposition is nasty.

(d) Define $f : \mathbb{C}^+ \rightarrow \mathbb{C}$ via

$$f(z) := \left\langle \delta_0, (-\Delta - z\mathbb{1})^{-1} \delta_0 \right\rangle.$$

Find an explicit expression for f using the Fourier series.

(e) Calculate the limit

$$\lim_{\varepsilon \rightarrow 0^+} \Im \{ f(E + i\varepsilon) \}$$

for the two cases $E \in (0, 4)$ and $E \in \mathbb{R} \setminus (0, 4)$.

(f) Calculate the spectral measure of the duo $(-\Delta, \delta_0)$ and determine its type (w.r.t. the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

7. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and on it define the multiplication operator

$$V(X)$$

via

$$(V(X)\psi)(x) := V(x)\psi(x) \quad (x \in \mathbb{Z}, \psi \in \ell^2(\mathbb{Z}))$$

where $V : \mathbb{Z} \rightarrow \mathbb{R}$ is some bounded sequence.

(a) For $x \in \mathbb{Z}$, is δ_x is a cyclic vector for $V(X)$?

(b) For any $x \in \mathbb{Z}$, define $f_x : \mathbb{C}^+ \rightarrow \mathbb{C}$ via

$$f_x(z) := \left\langle \delta_x, (V(X) - z\mathbb{1})^{-1} \delta_x \right\rangle.$$

(c) Calculate both

$$\lim_{\varepsilon \rightarrow 0^+} \Im \{ f_x(E + i\varepsilon) \}$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon \Im \{ f_x(E + i\varepsilon) \}$$

for all $E \in \mathbb{R}$ (separate into cases).

(d) Calculate the spectral measure of $(V(X), \delta_0)$ and determine its type (w.r.t. the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

(e) Define

$$I_\lambda := \{ n \in \mathbb{Z} \mid V(n) = \lambda \}$$

and

$$\mathcal{H}_\lambda := \text{span}(\{ \delta_n \}_{n \in I_\lambda}).$$

Show that

$$\ell^2(\mathbb{Z}) = \oplus_\lambda \mathcal{H}_\lambda$$

Find the cyclic vectors of $V(X)$ within each (possible direct sum of cyclic subspace of) $\mathcal{H}_\lambda$.

8. On $\ell^2(\mathbb{Z})$, consider $H = -\Delta + V(X)$ where $V : \mathbb{Z} \rightarrow \mathbb{R}$ obeys $V \rightarrow \infty$ as $|x| \rightarrow \infty$. Show that H has a cyclic vector.
9. On $\ell^2(\mathbb{N})$, consider $H = -\Delta + V(X)$ where $V : \mathbb{N} \rightarrow \mathbb{R}$ is general and $-\Delta$ is taken with Dirichlet boundary conditions. Show that H has a cyclic vector.
10. [Brownian motion and Brownian bridge] On $L^2([0, 1])$, define two integral operators

$$\begin{aligned} (Tf)(x) &:= \int_0^1 \min(\{x, y\}) f(y) dy \\ (Bf)(x) &:= \int_0^1 (\min(\{x, y\}) - xy) f(y) dy. \end{aligned}$$

These are the covariance operators of Brownian motion and the Brownian bridge respectively.

(a) Show that

$$\begin{aligned} \|T\|_{\text{HS}}^2 &= \int_{(x,y) \in [0,1]^2} |\min(\{x, y\})|^2 dx dy \\ \|B\|_{\text{HS}}^2 &= \int_{(x,y) \in [0,1]^2} |\min(\{x, y\}) - xy|^2 dx dy. \end{aligned}$$

Conclude the two operators are compact.

- (b) Find the non-zero eigenvalues and corresponding eigenfunctions of T and B . Compare them with those of the Laplacian (with two different boundary conditions) on $[0, 1]$.
- (c) Show that T and B are trace class and calculate their traces by summing up the non-zero singular values.
- (d) Compare with

$$\int_0^1 \min(\{x, x\}) dx, \int_0^1 (\min(\{x, x\}) - x^2) dx.$$