

MAT520 HW1

September 11, 2025

1. See [Q1](#).
3. See [Q2](#).
4. See [Q11](#).
5. See [Q13](#).
8. If $\|f_n - f\|_\infty \rightarrow 0$, then clearly $\sup_{x \in K} |f_n(x) - f(x)| \rightarrow 0$. Thus compact open topology (uniform convergence on compact sets) on C_b is strictly weaker than the sup-norm topology. To show that it is strictly weaker, consider a function f supported on $[0, 1]$ with $\|f\|_\infty = 1$ and $f_n(x) = f(x - n)$. Then $f_n(x) \rightarrow 0$ on any compact subsets of $[0, \infty)$. However f_n is not convergent in sup-norm topology.
12. It is clear that it is an equivalent relation, and hence it suffices to show that any norm is equivalent to the canonical one $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. After normalization, we need to show that there are $a, b > 0$ such that $a \leq \|x\| \leq b$ for all $x \in \mathbb{C}^n$ such that $\|x\|_2 = 1$. Observe that $K = \{x \in \mathbb{C}^n : \|x\|_2 = 1\}$ is compact with respect to $\|\cdot\|_2$. If $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$ is continuous with respect to $\|\cdot\|_2$, then $\|\cdot\|$ achieves maximum b and minimum a on K , which cannot be zero, since $\|x\| = 0$ implies that $x = 0 \in K$. By writing elements in $\mathbb{C}^n$ using the standard basis of $\mathbb{C}^n$, it is not hard to see that $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$ is indeed continuous with respect to $\|\cdot\|_2$.
13. Let $T : (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1)$ be the identity map. Then T is bijective and continuous since $\|Tx\|_2 = \|x\|_2 \leq C\|x\|_1$. Thus we can use the inverse mapping theorem to conclude.
16. See Lemma 3.18 in LN.
17. See Lemma 3.19 in LN.
19. The only if part is clear. To show the other direction, let $\{x_n\}$ be Cauchy. There exists a subsequence $\{x_{n_k}\}$ so that $\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}$. Let $y_k = x_{n_{k+1}} - x_{n_k}$. We

have $\sum_k \|y_k\| \leq \sum_k 2^{-k} < \infty$. Thus $\sum_k y_k$ converges. Since $\sum_{k=1}^l y_k = x_{n_{l+1}} - x_{n_1}$, it follows that x_{n_k} converges to some x . Since $\{x_n\}$ is Cauchy, it is clear that x_n also converges to x .

21. See [this](#) for discussion of product topology. If $f_n \rightarrow f$ in ℓ^p , then clearly $f_n(m) \rightarrow f(m)$ for each $m \in \mathbb{N}$. Thus product topology is weaker than ℓ^p -norm. To show that it is strictly weaker, consider the sequence $f_n(m) = 0$ for $m \neq n$ and 1 if $m = n$. Then $f_n \rightarrow 0$ point-wise, but $\|f_n\|_p = 1$ for all p .
22. Let $p \in [1, \infty)$. If $\{x_n\} \in l^p$, then $\sum_n |x_n|^p < \infty$, and hence $\sum_{n>N} |x_n|^p < \varepsilon$. Thus we can use $y_n = x_n$ for $n \leq N$ and $y_n = 0$ for $n > N$ to construct an approximating sequence. For $p = \infty$, consider the sequence $x_n = 1$ for all n . Then for any $y \in X$ we have $\|y - x\|_\infty = 1$.
24. See Example 3.10 in LN.
27. Consider

$$\|C_a b\|_1 := \sum_n \left| \sum_k a_{n-k} b_k \right| \leq \sum_n \sum_k |a_{n-k}| |b_k| \leq \sum_k |b_k| \sum_n |a_{n-k}| = \|a\|_1 \|b\|_1.$$

29. Let us show Y is closed. Let $f_n \in Y$ and $f_n \rightarrow f \in L^p$. Then

$$\left| \int_0^1 f \right| = \left| \int f - f_n + f_n \right| \leq \left| \int f - f_n \right| + \left| \int f_n \right| \leq \int_0^1 |f - f_n| \leq \|f - f_n\|_p \rightarrow 0.$$

30. Let $C_b(\mathbb{R} \rightarrow \mathbb{C})$ be the sup-norm metric space. We say that a set F is a relatively compact subsets of C_b if the closure of F is compact. We claim that F is characterized as: for any sequence in F there exists a subsequence that has a limit in C_b . The proof should be straightforward, using the fact that compact sets in C_b are the same as sequentially compact sets.