

Functional Analysis
Princeton University MAT520
HW2, Assigned on Sep 13th 2025

September 13, 2025

1 Baire Category

1. Show that the $\frac{1}{3}$ -Cantor set is nowhere dense.
2. Prove that if X is a *locally compact Hausdorff topological space* and $\{V_j\}_{j \in \mathbb{N}}$ are open dense sets then $\bigcap_{j \in \mathbb{N}} V_j$ is dense in X itself.
3. [extra] Show that $[0, 1]^{\mathbb{R}}$ is locally compact (via Tychonoff) but not metrizable. Show that $\mathbb{R} \setminus \mathbb{Q}$ with the Euclidean topology is metrizable but not locally compact.

Definition 1.1. A set $S \subseteq X$ is a Hamel basis for a Banach space X iff S is a maximal linearly independent subset of X , iff any $x \in X$ has a unique representation as a *finite* linear combination of elements of S .

4. Show that if X is an infinite-dimensional Banach space which is the union of countably many finite-dimensional subspaces, then X is of Baire's first category. Conclude that no infinite-dimensional Banach space has a countable Hamel basis.
5. Find a subset $S \subseteq [0, 1]$ which is of Baire's first category but whose Lebesgue measure equals 1.
6. For any $f \in L^2(\mathbb{S}^1)$, let $\hat{f} : \mathbb{Z} \rightarrow \mathbb{C}$ be given by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\theta \in [-\pi, \pi]} f(e^{i\theta}) e^{-in\theta} d\theta \quad (n \in \mathbb{Z}).$$

Define $\Lambda_n : L^2(\mathbb{S}^1) \rightarrow \mathbb{C}$ via

$$\Lambda_n f := \sum_{k=-n}^n \hat{f}(k) \quad (f \in L^2(\mathbb{S}^1))$$

Show that

$$\left\{ f \in L^2(\mathbb{S}^1) \mid \lim_{n \rightarrow \infty} \Lambda_n f \exists \right\}$$

is a dense subspace of $L^2(\mathbb{S}^1)$ of Baire's first category.

7. Let X be a Banach space and Y is a linear subspace of X whose complement is of Baire's first category. Show that $Y = X$.
8. Let X, K be metric spaces with K compact. Assume that $f : X \rightarrow K$ is a map with $\Gamma(f) \in \text{Closed}(X \times K)$. Show that f is continuous.
9. [extra] The Banach-Schauder theorem: Let $A : X \rightarrow Y$ be a bounded linear function between Banach spaces. Then show that only one of the possible two alternatives may hold:
 - (a) $AU \in \text{Open}(AX)$ for any $U \in \text{Open}(X)$, or,
 - (b) AX is of Baire's first category.

2 Open Mapping

10. Let $L : \mathcal{B}(\ell^2(\mathbb{N} \rightarrow \mathbb{C}))$ be the left shift operator

$$(La)_n := a_{n+1} \quad (n \in \mathbb{N}_{\geq 1}, a \in \ell^2(\mathbb{N} \rightarrow \mathbb{C})) .$$

Show that T is surjective and conclude it is an open map via the open mapping theorem. Also independently explicitly compute the image of the open unit ball under L and show it is open.

11. Assume $A : X \rightarrow Y$ is a bijective bounded linear operator between Banach spaces. Use the open mapping theorem to show that $A^{-1} : Y \rightarrow X$ is bounded.
12. Give an example of a surjective bounded linear operator between normed spaces (which are *not* Banach) that is *not* open. E.g. think of the right shift operator on the space of sequences of finite support.
13. Let X be a Banach space and $Y \subseteq X$ be a closed linear subspace. Let

$$\pi : X \rightarrow X/Y$$

be the quotient map. Show that π is open via the open mapping theorem.

14. [Perturbation theory] Let $A : X \rightarrow Y$ be a surjective bounded linear operator between Banach spaces. Show that there exists some $\delta > 0$ so that for all bounded linear operators of the form $B : X \rightarrow Y$ which obey

$$\|A - B\|_{\text{op}} < \delta$$

B is also surjective (and hence open).

15. Let $A : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ be given by

$$(Aa)_n := \frac{1}{n} a_n \quad (n \in \mathbb{N}_{\geq 1}, a \in \ell^2) .$$

- (a) Show that A is bounded and calculate $\|A\|_{\text{op}}$.
- (b) Show that A is injective but not surjective (you may use open mapping theorem for the latter, or work directly).