

MAT520 HW6

November 28, 2025

5. See Rudin Theorem 12.4.

6. See Rudin Theorem 12.6.

7. For a counterexample, consider $\mathcal{H} = \ell^2(\mathbb{N})$, take e_n the standard basis, let $\phi_0 = e_0$, and $\phi_n = e_n - e_{n-1}$ for $n > 0$. Then $\sum_{n=0}^N \phi_n = e_N$ does not converge, but for any ψ , we have

$$\sum_{n=0}^N \langle \psi, \phi_n \rangle = \langle \psi, e_N \rangle.$$

Since $\psi \in \ell^2(\mathbb{N})$, its terms tend to zero, and so the sum converges to 0 for large N .

8. For the sum, we expand this as

$$\frac{1}{N} \sum_{n=1}^N \alpha^n \langle \phi + \alpha^n \psi, \phi + \alpha^n \psi \rangle = (\|\phi\|^2 + \|\psi\|^2) \frac{1}{N} \left(\sum_{n=1}^N \alpha^n \right) + \langle \phi, \psi \rangle \frac{1}{N} \sum_{n=1}^N \alpha^{2n} + \langle \psi, \phi \rangle.$$

The first two sums vanish by the formula for a sum of a geometric series. Take $N \rightarrow \infty$ we get the integral equality.

9. Using Cauchy-Schwartz $|\langle \varphi_n, \psi_n \rangle| \leq \|\varphi_n\| \|\psi_n\|$ we see that $\|\varphi_n\|, \|\psi_n\| \rightarrow 1$. Then

$$\|\varphi_n - \psi_n\|^2 = \|\varphi_n\|^2 + \|\psi_n\|^2 - \langle \varphi_n, \psi_n \rangle - \langle \psi_n, \varphi_n \rangle \rightarrow 0.$$

10. We may assume $\|\phi\| \neq 0$ since otherwise norm convergence forces $\phi_n \rightarrow 0 = \phi$, and so for large n we may assume $\phi_n \neq 0$. Since $\lim \|\phi_n\| = \|\phi\|$, we may renormalize the sequence by dividing ϕ by $\|\phi\|$ and ϕ_n by $\|\phi_n\|$. Using the previous exercise and take its ψ_n to be ϕ , it then suffices to show $\langle \phi, \phi_n \rangle \rightarrow 1$. But this follows by taking $\xi = \phi$ for weak convergence.

11. Let $\beta_n = \langle \varphi_n, \psi \rangle$, and consider

$$F(\alpha_1 + \beta_1, \dots, \alpha_N + \beta_N) = \|\psi - \sum \beta_n \varphi_n - \sum \alpha_n \varphi_n\| = \sqrt{\|\xi\|^2 + \sum |\alpha_n|^2}$$

where $\xi_n = \psi - \sum \beta_n \varphi_n$ and the last equality follows from orthogonality; this is clearly minimized for $\alpha_n = 0$.