

MAT520 HW7

November 28, 2025

1. We have

$$|(a - aa^*a)^*|^2 = (a - aa^*a)(a^* - a^*aa^*) = |a|^2 - 2(|a|^2)^2 + (|a|^2)^3$$

but this vanishes since $|a|^2$ is idempotent. By C-star identity we have $a - aa^*a = 0$.

2. If a is a partial isometry, then

$$(|a^*|^2)^2 = (aa^*a)a^* = aa^*$$

where we use the previous exercise.

3. Since p, q are self-adjoint projections, they have spectra $\{0, 1\}$. Then $p - 1/2, q - 1/2$ are self-adjoint with spectral radius $1/2$, and hence $\|p - 1/2\|, \|q - 1/2\| = 1/2$. Thus

$$\|p - q\| = \|(p - 1/2) - (q - 1/2)\| \leq 1/2 + 1/2 = 1.$$

4. If u, v are unitary then

$$\|u - v\|^2 = \|2 - uv^* - u^*v\| \leq 2 + \|uv^*\| + \|u^*v\| \leq 4$$

since unitaries have norm 1.

6. For (a) implies (b), we have

$$(p + q)^2 = p + pq + qp + q = p + q$$

since $qp = (pq)^* = 0$. (b) implies (c) is immediate as $1 - p - q$ is also a projection. For (c) implies (a), we have $p + q \leq 1$, and $p \leq 1 - q$, we may multiply on left and right by q and get

$$(pq)^*(pq) = qpq \leq q(1 - q)q = 0.$$

Thus $pq = 0$.

10. Using spectral mapping theorem, we have $\sigma(p^2 - p) = \{\lambda^2 - \lambda : \lambda \in \sigma(p)\} = \{0\}$. Thus $\lambda^2 - \lambda = 0$ for $\lambda \in \sigma(p)$.
11. Use previous exercise and Theorem 8.6 in LN.
12. Theorem 8.6 in LN.
13. Lemma 8.5 in LN.
14. Theorem 8.15 in LN.
15. Claim 8.9 in LN.
16. The hypothesis implies that $|a|^2, |a^*|^2$ are invertible. a has left inverse $((a^*a)^{-1}a^*)a$, and right inverse $aa^*(aa^*)^{-1}$. Thus a is invertible.