

MAT520 HW8

December 7, 2025

1. See Lemma 9.13 in LN.
2. (a) Since $\langle R\phi, \psi \rangle = \sum_{i \in \mathbb{N}} \overline{\phi_{i+1}} \psi_i$, we see $L := R^*$ is the unilateral left shift operator $(L\psi)_{n+1} = \psi_n, (L\psi)_0 = 0$.
 (b) We have $|R|^2 = 1$, while $|L|^2$ is the orthogonal projection onto $\text{span}(\psi_n)_{n>0}$.
 (c) The identity is an idempotent, as is an orthogonal projection.
 (d) We have seen from the spectral mapping theorem that a nontrivial idempotent has spectrum $\{0, 1\}$ while 1 has spectrum $\{1\}$. This gives $\sigma(|R|^2) = \{1\}$ and $\sigma(|L|^2) = \{0, 1\}$. For R , we see that $[(R - \lambda)\phi]_i = \phi_{i+1} - \lambda\phi_i$. If $|\lambda| < 1$ we see the geometric series $\phi_i = \lambda^i$ is annihilated, and so $\sigma(R)$ contains $B_1(0)$. Since it is also compact, it therefore also contains the closed unit ball. And since $\|R\| = 1$, we have that $r(R) \leq 1$, and so $\sigma(R)$ is exactly the closed unit ball.

For L we similarly have $\sigma(L)$ inside the closed unit ball since $\|L^n\| = 1$ for all n , and then for $|\lambda| < 1$ we have $[(L - \lambda)\phi]_i = \phi_{i-1} - \lambda\phi_i$. This has trivial kernel except for $\lambda = 0$ since the necessary geometric series of ratio $1/\lambda$ would not converge. As for $\lambda \neq 0$, consider the system $(L - \lambda)\phi = \psi$, or $\phi_i = (\phi_{i-1} - \psi_i)/\lambda, \phi_0 = -\psi_0/\lambda$. By induction we see

$$\phi_k = - \sum_{i=0}^k \frac{\psi_i}{\lambda^{k+1-i}}.$$

Since $|\lambda| < 1$ the resulting series is clearly unsummable for, say, $\psi_i = 1/i^2$. Hence all such λ are part of the continuous spectrum, and we again obtain the entire closed unit ball as spectrum.

3. (a) This is clearly $\hat{L} := (\hat{L}\psi)_n = \psi_{n-1}$.
 (b) These are both the identity.
 (c) Obvious from (b).
 (d) Both $|\hat{L}|^2$ and $|\hat{R}|^2$ are the identity, and so have spectrum $\{1\}$. Then $\hat{L}$ and $\hat{R}$ differ only by the coordinate relabelling $e_n \mapsto e_{-n}$ taking λI to itself, and so will

have the same spectrum. We see also that $\hat{R}$ is unitary, and so this spectrum lies inside S^1 . Then consider $|\lambda| = 1$ and $(\hat{R} - \lambda)\phi = \psi$, so that we have for all n that $\psi_n = \phi_{n+1} - \lambda\phi_n$. Take $\phi_0 = a$. Then for $k > 0$ we have

$$\phi_k = - \sum_{i=0}^k \frac{\psi'_i}{\lambda^{k+1-i}}$$

where $\psi'_i = \psi_i$ for $i > 0$, a for $i = 0$. For $\psi_i = \lambda^{-i}/(|i| + 1)$, we obtain constructive interference and so the ϕ_k do not even tend to zero. Hence all points in S^1 lie in the spectrum. The same is true of $\hat{L}$.

4. (a) This operator is clearly self-adjoint since $1/j$ is real.
 - (b) We see $(1/X) - \lambda I$ is invertible if and only if $(-\lambda + 1/j)^{-1}$ exists for all j and remains bounded. For $\lambda \neq 0$, this has modulus $1/|\lambda|$ in the limit, and so only $\lambda = 1/j$ belongs to the spectrum for nonzero λ . As for $\lambda = 0$ the resulting inverse is not a bounded operator, so $\sigma(1/X) = \{0\} \cup \{1/j\}$.
 - (c) 0 is not isolated, so this follows from Lemma 7.19.
5. Let $x = a + b, y = c + d$. Then

$$\langle P_M x, y \rangle = \langle a, c + d \rangle = \langle a, c \rangle = \langle a + b, c \rangle = \langle x, P_M y \rangle,$$

since a, c are orthogonal to c, d . Hence P_M is self-adjoint, and $P_M^2 = P_M$ is clear. Conversely, for such an operator P take $M = \text{im } P$. We see by self-adjointness that $\ker P \subset M^\perp$, as

$$\langle Px, y \rangle = \langle x, Py \rangle = \langle x, 0 \rangle = 0$$

for y in the kernel. Furthermore, M and $\ker P$ span since $P(x - Px) = Px - P^2x = 0$, and so $x = Px + (x - Px)$ gives a decomposition. Hence P is the projection onto its image.

9. Let $\psi \in \mathcal{H}$. Then

$$\|A_n B_n \psi - AB\psi\| = \|A_n B_n \psi - A_n B \psi + A_n B \psi - AB\psi\| \leq \|A_n(B_n - B)\psi\| + \|A_n B \psi - AB\psi\|.$$

The second term becomes small since $A_n \rightarrow A$ in the strong operator topology and $B\psi$ is fixed. As for the first term, since $(B_n - B)\psi$ becomes small it suffices to show the A_n are bounded in norm. Otherwise, let ψ_0 be any nonzero vector. At stage k after selecting operators $A_{n_1}, \dots, A_{n_{k-1}}$ and a vector ψ_{k-1} we may go out to some operator A_{n_k} of norm at least $e^{k^2} 2^k \|\psi_{k-1}\| \max_{i < k} (\|A\|, \|A_{n_i}\|)$, and perturb ψ_{k-1} by a length $(2^k \|\psi_k\| \max_{i < k} (\|A\|, \|A_{n_i}\|))^{-1}$ vector that gets mapped to a vector of length at least e^{k^2} by A_{n_k} . Since this contributes only a small amount to the norm of $A_{n_j} \psi_k$ for $j < k$, and these clearly converge to some ψ , we obtain a ψ with $\|A_{n_k} \psi\| \rightarrow \infty$, contradicting strong convergence.

12. First we show that when $A_n \rightarrow A$ either strongly or in norm, A is positive. We have that for any $\psi \in \mathcal{H}$,

$$\langle \psi, A\psi \rangle = \langle \psi, A_n\psi \rangle + \langle \psi, (A - A_n)\psi \rangle \geq \langle \psi, A_n\psi \rangle - |\langle \psi, (A - A_n)\psi \rangle|,$$

where in the last step we used the fact that A_n is positive, so $\langle \psi, A_n\psi \rangle \geq 0$. As $A_n \rightarrow A$ either in norm or strongly, we have that $|\langle \psi, (A - A_n)\psi \rangle| \rightarrow 0$, so $\langle \psi, A\psi \rangle \geq 0$. Thus, A is positive.

Now assume that $A_n \rightarrow A$ in norm.

We claim that for any polynomial p , if $A_n \rightarrow A$ in norm, then $p(A_n) \rightarrow p(A)$ in norm. Indeed, since $A_n \rightarrow A$ in norm, we have that $A_n \rightarrow A$ strongly, and in Exercise 25 we have shown that this implies that $\sup_n \|A_n\| < \infty$. Set M to be the maximum of $\sup_n \|A_n\|$ and $\|A\|$. We have that for any positive integer m ,

$$\begin{aligned} \|A_n^m - A^m\| &\leq \sum_{k=1}^m \|A_n^{m-k+1}A^{k-1} - A_n^{m-k}A^k\| \\ &\leq \sum_{k=1}^m \|A_n\|^{m-k} \|A\|^{k-1} \|A_n - A\| \\ &\leq M^k \cdot m \cdot \|A_n - A\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. By taking linear combinations, we have that $p(A_n) \rightarrow p(A)$ in norm. $\square$

By continuity of the spectrum and the fact that A and the A_n are positive, we have that $\sigma(A) \cup \bigcup_n \sigma(A_n) \subseteq [0, R]$ for some $R > 0$. Let $f(x) = \sqrt{x}$. By Stone–Weierstrass, there exist polynomials p_n such that $p_n \rightarrow f$ uniformly on $[0, R]$. By continuous functional calculus, cf. Reed–Simon VII.1 property (b), for any self-adjoint operator B ,

$$\|f(B) - p_n(B)\| \leq C_B \|f - p_n\|_\infty$$

for some constant $C_B > 0$. Thus,

$$\begin{aligned} \|f(A_n) - f(A)\| &\leq \|f(A_n) - p_k(A_n)\| + \|p_k(A_n) - p_k(A)\| + \|p_k(A) - f(A)\| \\ &\leq (C_{A_n} + C_A) \|f - p_k\|_\infty + \|p_k(A_n) - p_k(A)\|. \end{aligned}$$

By the above claim, as $n \rightarrow \infty$, we have $\|p_k(A_n) - p_k(A)\| \rightarrow 0$, and as $k \rightarrow \infty$, we have $\|f - p_k\|_\infty \rightarrow 0$. Thus, $\|f(A_n) - f(A)\| \rightarrow 0$ as $n \rightarrow \infty$, i.e. $\sqrt{A_n} \rightarrow \sqrt{A}$ in norm.

Note that this argument goes through by replacing absolute values with pointwise evaluation, so $A_n \rightarrow A$ strongly implies $\sqrt{A_n} \rightarrow \sqrt{A}$ strongly.

17. Let $\psi \in \mathcal{H}$. Then, we have $a_n = \|(PQ)^n\psi\|$ forms a monotonically decreasing sequence, since P, Q are projection maps and so must decrease norm. The a_n are also bounded

below by 0, and so have some limit L . Let $\epsilon > 0$ and let N be so large that $a_N < L + \epsilon$. Then take $\phi = (PQ)^N \psi$ and consider $(PQ)^k \phi$ for $k > 0$. At each stage applying Q , we break up the current vector φ as $\varphi_N + \varphi_{N^\perp}$ via the decomposition $\mathcal{H} = N \oplus N^\perp$. Then $\|\varphi\|^2 = \|\varphi_N\|^2 + \|\varphi_{N^\perp}\|^2$ by orthogonality, and $\|\varphi_N\| \geq L$, so that $\|\varphi_{N^\perp}\|^2 \leq 2L\epsilon + \epsilon^2$. Applying P to φ_N then produces a similar orthogonal decomposition, and applying $(PQ)^k$ in this manner to ϕ , we obtain an orthogonal decomposition via vectors in $N^\perp, M^\perp$ whose norms squared all sum to at most $2L\epsilon + \epsilon^2$. Hence $\phi - (PQ)^k \phi$, which is the sum of these $N^\perp, M^\perp$ components, has norm at most $\sqrt{2L\epsilon + \epsilon^2}$. This is independent of k going to zero as $\epsilon \rightarrow 0$, and so $(PQ)^n \psi$ converges in norm. In particular, the strong limit of $(PQ)^n$ exists.

Call this strong limit S . Then by construction we have $PS = S$, and by considering the intermediate evaluations of Q handled in the norm bound we have also $QS = S$, as S is equivalently the strong limit of $Q(PQ)^n$. Hence S maps into $M \cap N$. It also clearly fixes all elements of $M \cap N$, so this will be the subspace it projects onto if it is an orthogonal projection. S^2 will clearly be the strong limit of $(PQ)^{2n}$, which is simply S , and so we need only show S is self-adjoint.

For this, we see S^* will be the strong limit of $(QP)^n$ as the termwise adjoint of $(PQ)^n \rightarrow S$, and we similarly have $PS^* = S^*$. But $PS^* = SP$ since $P(QP)^n = (PQ)^n P$, and so it suffices to show $SP = S$. Then letting $\psi = \psi_M + \psi_{M^\perp}$, we have $SP\psi = S\psi_M$ and $S\psi = S\psi_M + S\psi_{M^\perp}$, and so we wish to show S annihilates $M^\perp$. We see however that S^* will also map to $M \cap N$, and so we have for $\phi \in M^\perp$ that

$$\|S\phi\|^2 = \langle S\phi, S\phi \rangle = \langle \phi, S^* S\phi \rangle = 0,$$

since $S^* S\phi \in M \cap N \subset M = (M^\perp)^\perp$. Hence $S\phi = 0$. So, $S^2 = S = SP = PS^* = S^*$ and we conclude S is the self-adjoint projection onto $M \cap N$.

20. Since conjugation commutes with inversion, we have by the continuous functional calculus and $A = A^*$ that $((A - z1)^{-1})^* = (A - \bar{z}1)^{-1}$. Then

$$(A - z1)^{-1}(A - \bar{z}1)^{-1} = (A^2 - 2\operatorname{Re}(z)A + |z|^2 1)^{-1}.$$

Letting $z = x + iy$, we have $|z|^2 = x^2 + y^2$, so this becomes $((A - x1)^2 + y^2 1)^{-1}$. By spectral mapping and $\sigma(A) \subset \mathbb{R}$, the inside term has spectrum lying in $[y^2, \infty)$. Thus its inverse has spectrum in $[0, 1/y^2]$. It is also self-adjoint, so its norm is at most $1/y^2$. By the C-star identity we then have

$$\|(A - z1)^{-1}\|^2 = \|((A - z1)^{-1})^*(A - z1)^{-1}\| \leq 1/y^2.$$

Taking square roots gives the desired.

22. (a) Clearly V is defined and linear by linearity of the integral. As for boundedness, we have

$$\langle V(\psi), V(\psi) \rangle \leq \int_0^1 \left(\int_0^x \int_0^x |\psi(x)\psi(y)| d\mu \right) \leq \int_0^1 \left(\int_0^x \int_0^x |\psi(x)|^2 + |\psi(y)|^2 d\mu \right) d\mu$$

by AM-GM. For each $x \in [0, 1]$ the inner integral is at most $2\|\psi\|^2$, so we see $\|V\| \leq \sqrt{2}$. As for V^* , we have

$$\langle V\psi, \phi \rangle = \int_0^1 \int_0^x \psi(y) \overline{\phi(x)} d\mu_y d\mu_x,$$

and by swapping from vertical to horizontal integration we obtain

$$\int_0^1 \int_y^1 \psi(y) \overline{\phi(x)} d\mu_x d\mu_y = \langle \psi, V^* \phi \rangle.$$

Hence V^* is as desired.

- (b) First note that for $f \in L^2([0, 1])$, we may approximate f arbitrarily close in norm by bounded functions via truncation, which may then be approximated by step functions, followed by step functions on intervals, which are easily approximated by continuous functions. These are well-approximated in L^2 by $e_n = e^{2\pi i n x}$ by Fourier analysis, and so these give an orthonormal basis for L^2 . Then we note for $n \neq 0$ that

$$(Ve_n)(x) = \int_0^x e^{2\pi i n y} dy = -\frac{i}{2\pi n} (e_n(x) - 1).$$

Then $Ve_n, n \neq 0$ has norm $\frac{1}{2\pi n} \|e_n - 1\| = \frac{1}{\sqrt{2\pi n}}$. We then have

$$V = \lim \sum_{-n}^n Ve_n \otimes e_n^*,$$

which converges in norm by the triangle inequality together with the above computation. This is a norm limit of finite rank operators, and thus $V - (Ve_0 \otimes e_0^*)$ is a compact operator. Of course $Ve_0 \otimes e_0^*$ is itself compact as a rank one operator, so V is compact as the sum of two compact operators, which may be seen by condition 2 of Lemma 9.33.

Hence its spectrum is only point spectrum and potentially zero. We therefore reduce to showing V has no nonzero eigenvalues, since then $\sigma(V)$ is a nonempty subset of $\{0\}$, giving $\sigma(V) = \{0\}$ and $r(V) = 0$ immediately. Then if $\lambda \neq 0$, consider the equation

$$\lambda\psi(x) = \int_{[0, x]} \psi(y) d\mu.$$

The right hand side is differentiable for $\psi \in L^2([0, 1])$ and so we may extract the differential equation

$$\begin{aligned}\lambda\psi' &= \psi \\ \psi(x) &= Ae^{x/\lambda}.\end{aligned}$$

But since $(V\psi)(0) = 0$ for all ψ , such an eigenfunction cannot exist.

- (c) We compute $\|V^*V\|$. V^* is compact similarly to V , so that the composition V^*V is compact. It is also self-adjoint, and norm is equal to spectral radius in this case we reduce to finding eigenvalues of the compact V^*V to calculate its norm $\|V\|^2$. Then consider

$$\lambda\psi(x) = \int_x^1 \left(\int_0^y \psi(s)ds \right) dy.$$

Differentiating gives

$$\lambda\psi'(x) = - \int_0^x \psi(s)ds,$$

and differentiating again gives

$$\lambda\psi'' = -\psi.$$

We face initial conditions $\psi(1) = 0, \psi'(0) = 0$ from the integral bounds with ψ some sin or cos function. One sees that $A \cos((2n+1)\pi x/2)$ is the general form of an eigenvector, and so $\lambda = 4/(\pi^2(2n+1)^2)$. The largest such λ is from the choice $n = 0, -1$, so that $r(V^*V) = 4/\pi^2 = \|V\|^2$, or $\|V\| = 2/\pi$ as desired.