

MAT520 HW9

December 7, 2025

1. Let $\{e_n\}$ be an ONB, then

$$\sqrt{2} = \|e_n - e_m\| = \|A^{-1}A(e_n - e_m)\| \leq \|A^{-1}\| \|A(e_n - e_m)\|$$

Thus $\{Ae_n\}$ cannot have convergent subsequence. We use Lemma 9.35 in LN to reach a contradiction.

2. See Lemma 9.36 in LN.

3. See Theorem 9.42 in LN.

4. (a) $u \otimes v^*$ is finite-rank; thus compact.

(b) See pp. 190 in Stein and Shakarchi's Real analysis: measure theory, integration, and Hi.

- (c) Consider the finite-rank operator

$$A_N = \sum_{n=1}^N \frac{1}{1+n^2} e_n \otimes e_n^*$$

Then A_N converges to A in norm and hence A is compact.

5. $|H|^2$ is given by

$$|H|^2 = H^2 = \begin{bmatrix} |S|^2 & 0 \\ 0 & |S^*|^2 \end{bmatrix},$$

so that $|H| = |S| \oplus |S^*|$. As for determining U , let V, V' be the operators in the polar decompositions of S, S^* respectively. Then clearly

$$U = \begin{bmatrix} 0 & V' \\ V & 0 \end{bmatrix}$$

satisfies $U|H| = H$. Its kernel is $\ker V \oplus \ker V' = \ker S \oplus \ker S^* = \ker H$, and its adjoint is

$$U^* = \begin{bmatrix} 0 & V^* \\ (V')^* & 0 \end{bmatrix},$$

so that

$$|U|^2 = \begin{bmatrix} |V|^2 & 0 \\ 0 & |V'|^2 \end{bmatrix}.$$

This is idempotent as a direct sum of idempotents, and so U is a partial isometry as desired. In fact $V' = V^*$ as one can verify. See, e.g., pp. 400 of Kadison's Fundamentals of the Theory of Operator Algebras Volume II: Advanced Theory.

6. Since finite rank operators are compact, it suffices to show that if A is an idempotent which is not a finite rank operator, then A is not compact. First note that if $Ax_n \rightarrow y$ for some $x_n \in \mathcal{H}$ and $y \in \mathcal{H}$, then using the fact that A is continuous,

$$Ay = A \lim Ax_n = \lim A^2 x_n = \lim Ax_n = y,$$

so $\text{im } A$ is closed. Since A is not finite rank, $\text{im } A$ is infinite-dimensional Hilbert space, there exists an orthonormal set $\{A\varphi_n\}_{n \in \mathbb{N}}$ in $\text{im } A$ for some $\varphi_n \in \mathcal{H}$. This means that for $i \neq j$,

$$\|A\varphi_i - A\varphi_j\|^2 = \|A\varphi_i\|^2 + \|A\varphi_j\|^2 = 2,$$

so $\|A\varphi_i - A\varphi_j\| \geq \sqrt{2}$. If A is compact, since the sequence $\{A\varphi_n\}$ is bounded, the sequence $\{A^2\varphi_n\} = \{A\varphi_n\}$ would have a convergent subsequence. From $\|A\varphi_i - A\varphi_j\| \geq \sqrt{2}$ for $i \neq j$, the sequence $\{A\varphi_n\}$ cannot have a convergent subsequence. Therefore, A is not compact.

7. Since M_f is nonzero, f is not zero almost everywhere. Thus, there exists some $\epsilon > 0$ such that the set

$$B = \{x \in [0, 1] : |f(x)| \geq \epsilon\}$$

has positive measure, denoted by $\mu(B) > 0$. Since $\mu(B) > 0$, we can choose a sequence of pairwise disjoint subsets $\{B_n\}_{n=1}^\infty$ of B such that $0 < \mu(B_n) < \infty$ for all n .

We define the sequence of functions u_n by normalizing the characteristic functions of these sets:

$$u_n(x) = \frac{1}{\sqrt{\mu(B_n)}} \chi_{B_n}(x).$$

This sequence is orthonormal.

Consider the sequence $v_n = M_f u_n = f u_n$; the functions v_n and v_m are orthogonal. We have

$$\|v_n\|_2^2 = \int_{B_n} |f(x)|^2 |u_n(x)|^2 dx = \int_{B_n} |f(x)|^2 \frac{1}{\mu(B_n)} dx.$$

Since $|f(x)| \geq \epsilon$ for all $x \in B_n \subseteq B$, we have:

$$\|v_n\|_2^2 \geq \int_{B_n} \epsilon^2 \frac{1}{\mu(B_n)} dx = \epsilon^2.$$

Then

$$\|v_n - v_m\|_2^2 = \|v_n\|_2^2 + \|v_m\|_2^2 \geq \epsilon^2 + \epsilon^2 = 2\epsilon^2.$$

Thus $\{v_n\}$ cannot have convergent subsequence.

12. See Proposition 9.46 in LN.
14. See Claim 9.50 in LN.
15. Let A be compact and Fredholm. Using Atkinson's theorem, there exists B such that $\mathbb{1} - AB$ is compact. Since A is compact, it follows that $\mathbb{1}$ is compact, which contradicts the first exercise.
16. This follows from Theorem 9.55 in LN.