Lectures 14 and 15

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1 Rotations and angular momentum

1.1 Properties of rotation in 3D

Rotations act on 3D real vectors via $v \mapsto Rv$ where R is a 3×3 real matrix. Since rotations preserve scalar product, we get

$$u \cdot v = u^T v = u^T R^T R v \tag{1}$$

which holds for arbitrary vectors u and v. This implies that

$$R^T R = \mathbb{1} \tag{2}$$

which means that R is an orthogonal matrix. 3×3 Orthogonal matrices form a group called $\mathrm{O}(3)$. To see this, note that for $R_{1,2} \in \mathrm{O}(3)$, $(R_1 R_2)^T R_1 R_2 = R_2^T R_1^T R_1 R_2 = \mathbb{1}$ which implies that $R_1 R_2 \in \mathrm{O}(3)$ (the existence of an inverse and an identity are obvious). Note, however, that not any orthogonal matrix represents a rotation since condition (2) is also satisfied by mirror reflections. For example, $M_z = \mathrm{diag}(1, 1, -1)$ represents a reflection about the $x - y$ plane. To distinguish the two, we note that Eq. 2 implies that $\det R = \pm 1$ (since $\det R^T = \det R$). Reflections always have $\det R = -1$ whereas pure rotations always have $\det R = +1$. Thus, we can split the orthogonal group into elements with $\det R = +1$, which represent pure rotations, and elements with $\det R = -1$ which represent combinations of a reflection and a rotation. Note that only the first set of elements forms a group since the product of two matrices with determinant -1 yields a matrix with determinant $+1$. The group of orthogonal matrices with determinant $+1$ is called the special orthogonal group, denoted by $\mathrm{SO}(3)$.

A rotation in 3D space is specified by a rotation axis $\hat{n}$, that is left invariant under the rotation, and a rotation angle φ. We will denote such rotation by $R_{\hat{n}}(\varphi)$ and use the convention that φ is positive for counterclockwise rotations. This means that a rotation is specified by three real parameters since $\hat{n}$ specifies a point on the two-dimensional sphere. We can see that this is also the count we get from Eq. 2 whose right hand side RR^T is a symmetric matrix specified by 6 real parameters which represents 6 constraints leading to $9 - 6 = 3$ free parameters.

The eigenvalues of any matrix in $\mathrm{SO}(3)$ are $1, e^{i\varphi}$, $e^{-i\varphi}$ [1]. Given an element R of $\mathrm{SO}(3)$, we can find the rotation axis $\hat{n}$ as the eigenvector corresponding to eigenvalue 1 which means it is left invariant by R. The rotation angle is given by the phase φ of one of the nonzero eigenvalues $e^{\pm i\varphi}$ where the sign ambiguity can be fixed by acting on any specific vector in the plane perpendicular to $\hat{n}$ (vectors in that plane can be constructed as linear superpositions of the two nontrivial eigenvectors of R that yield a real vector). Note that there is a sign ambiguity in defining $\hat{n}$ and φ since $R_{-\hat{n}(-\varphi)} = R_{\hat{n}}(\varphi)$.

It is a known fact the rotations about different axes in 3D space do not commute. This is something you can see easily by considering some simple examples. For instance, $\pi/2$ rotation about the z-axis followed by $\pi/2$ rotation about the x-axis is not the same as doing the x-rotation before the z-rotation. For instance, the first sequence, which we can denote by $R_x(\pi/2) R_z(\pi/2)$ maps the north pole on the sphere to the point $(0, -1, 0)$ whereas the second sequence $R_z(\pi/2) R_x(\pi/2)$ maps it to $(1, 0, 0)$.

[1] To see this, note that the eigenvalues should have absolute value 1 and their product should be $+1$

To specify the commutation relations for general rotation operators, we notice that the operator

$$Q_{\hat{n},\hat{m}}(\theta,\varphi) = R_{\hat{m}}(\theta)^T R_{\hat{n}}(\varphi) R_{\hat{m}}(\theta) \tag{3}$$

leaves $R^T_{\hat{m}}(\theta)\hat{n}$ invariant and its eigenvalues are $1, e^{\pm i\varphi}$ which implies that $Q_{\hat{n},\hat{m}}(\theta,\varphi) = R_{R^T_{\hat{m}}(\theta)}(\varphi)$ [2]. As a result, we have the relation

$$R_{\hat{m}}(\theta)^T R_{\hat{n}}(\varphi) R_{\hat{m}}(\theta) = R_{R^T_{\hat{m}}(\theta)\hat{n}}(\varphi) \tag{4}$$

which holds for arbitrary unit vectors $\hat{n}$ and $\hat{m}$ and arbitrary angles θ and φ. You can verify that this relation holds in some simple cases. For example, if $\hat{n} = \hat{m}$, the two rotations commute and the RHS reduces to $R_{\hat{n}}(\varphi)$. The LHS also reduces to $R_{\hat{n}}(\varphi)$ since $R^T_{\hat{m}}(\theta)\hat{n} = \hat{n}$. A less trivial example is taking $\hat{m} = \hat{z}$, $\hat{n} = \hat{x}$ and $\theta = \varphi = \pi/2$. The LHS gives rotation around the y-axis by $\pi/2$. It is easy to verify this is the same we get by rotating by $\pi/2$ around $\hat{z}$, then by $\pi/2$ around $\hat{x}$, and then by $-\pi/2$ around $\hat{z}$.

Similar to our approach with different symmetry operations, we can gain a lot of insights by considering infinitesmal rotations $R_{\hat{n}}(d\varphi)$

$$R_{\hat{n}}(d\varphi) = \mathbb{1} + \Gamma_{\hat{n}} d\varphi \tag{5}$$

Substituting in Eq. 2, we find that

$$\Gamma^T_{\hat{n}} = -\Gamma_{\hat{n}} \tag{6}$$

An arbitrary real antisymmetric matrix can be expanded in terms of the three real antisymmetric matrices

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Thus, we can write $\Gamma_{\hat{n}} = \sum_{i=1}^3 \alpha^i_{\hat{n}} \gamma_i$. In the following, we will not write the summation explicitly and always assume that repeated indices are summed over. The condition $R_{\hat{n}}(\varphi) = \hat{n}$ for any φ implies

$$\Gamma_{\hat{n}}\hat{n} = 0 = \alpha^i_{\hat{n}}\gamma_i\hat{n} = \boldsymbol{\alpha}_{\hat{n}} \times \hat{n} \tag{8}$$

where $\boldsymbol{\alpha}$ is the vector whose components are α_i. The equation above means that $\boldsymbol{\alpha}_{\hat{n}}$ is parallel to $\hat{n}$, i.e. $\alpha^i_{\hat{n}} = \alpha\hat{n}^i$ for some constant α Since this constant is universal (independent of both $\hat{n}$ and φ), we can fix it by considering the rotation matrix for some simple rotation. For example, rotation around the z-axis is given by

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Expanding to leading order in φ gives

$$\Gamma_{\hat{z}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \gamma_3 \tag{10}$$

This means that the constant $\alpha = 1$ and we can write

$$\Gamma_{\hat{n}} = \sum_{i=1}^3 \hat{n}_i\gamma_i = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix} \tag{11}$$

Substituting in (5), we get

$$R_{\hat{n}}(d\varphi) = \mathbb{1} + d\varphi\hat{n}^i\gamma_i \tag{12}$$

The generators for the 3D rotation γ_i satisfying

$$[\gamma_i, \gamma_j] = \epsilon_{ijk}\gamma_k \tag{13}$$

We can then write a finite rotation by exponentiating the generator using

$$R_{\hat{n}}(\varphi) = \lim_{N\to\infty}(\mathbb{1} + \frac{\varphi}{N}\hat{n}^i\gamma_i)^N = e^{\varphi\hat{n}^i\gamma_i} \tag{14}$$

[2] This condition fixes this form up to a sign $\pm\varphi$. You can use any vector perpendicular to $\hat{n}$ to fix this sign to be the positive one

1.2 Representations of rotation

To describe the action of rotations on a quantum state specified by a ket $|\alpha\rangle$, we need to find a unitary operator that 'represents' the rotation in the sense that

$$|u\rangle_R = D(R)|u\rangle \tag{15}$$

We see that $D(R)$ has to satisfy several consistency conditions. First, for $R = R_1R_2$, we have $|u\rangle_R = |u\rangle_{R_1R_2}$ which implies $D(R) = D(R_1R_2) = D(R_1)D(R_2)$. We also see that $D(\mathbb{1}) = \mathbb{1}$ and $\mathbb{1} = D(RR^T) = D(R)D(R^T)$ which implies $D(R^T) = D(R)^{-1} = D(R)^\dagger$. This means that D defines a unitary representation of SO(3). It is important to emphasize the distinction between R which is a 3×3 real orthogonal matrix and $D(R)$ which is a unitary matrix whose dimension is the Hilbert space dimension and can be arbitrary. For example, for spin-$1/2$, $D(R)$ will be two-dimensional. Note also that when defining the properties of $D(R)$, we could have allowed for some arbitrary phases since $|u\rangle$ is defined up to a phase. These phases need to satisfy some consistency conditions. Such representations are called projective representations. For our current analysis, where we focus on infinitesmal rotations, we can restrict ourselves to the projective case since such phases drop.

We now introduce the infinitesmal generator for $D(R)$

$$D(R_{\hat{n}}(d\varphi)) = \mathbb{1} - iG_{\hat{n}}d\varphi = D(\mathbb{1} + d\varphi\hat{n}^i\gamma_i) = \mathbb{1} + d\varphi\hat{n}^i\frac{d}{d\epsilon}D(\mathbb{1} + \epsilon\gamma_i)|_{\epsilon=0} \tag{16}$$

This implies that $G_{\hat{n}}$ has the form

$$G_{\hat{n}} = \frac{J_i}{\hbar}\hat{n}^i, \qquad J_i = i\hbar\frac{d}{d\epsilon}D(\mathbb{1} + \epsilon\gamma_i)|_{\epsilon=0} \tag{17}$$

Previously, we have identified the generator of translation with momentum and the generator of time-translation with energy (the Hamiltonian). Both identification can be motivated by classical considerations. In classical mechanics, angular momentum is associated with the generators of rotation. This motivates us to identify J_i with angular momentum measured in units of $\hbar$.

Similar to what we did for R, we can construct the elements $D(R)$ corresponding to finite rotation by exponentiating the generator

$$D(R_{\hat{n}}(\varphi)) = e^{-i\varphi\frac{J_i}{\hbar}\hat{n}^i} = D(e^{\varphi\gamma_i\hat{n}^i}) \tag{18}$$

We are now ready to derive the very important angular momentum commutation relations in quantum mechanics. To do this, we write

$$e^{-i\epsilon_2\frac{J_j}{\hbar}}e^{-i\epsilon_1\frac{J_i}{\hbar}}e^{i\epsilon_2\frac{J_j}{\hbar}}e^{i\epsilon_1\frac{J_i}{\hbar}} = e^{-i\epsilon_2\frac{J_j}{\hbar}}e^{-\frac{i}{\hbar}\epsilon_1\mathrm{ad}_{J_i}}e^{-i\epsilon_2\frac{J_j}{\hbar}} = D(e^{\epsilon_2\gamma_j}e^{\epsilon_1\gamma_i}e^{-\epsilon_2\gamma_j}e^{-\epsilon_1\gamma_i}) = D(e^{\epsilon_2\gamma_j}e^{\epsilon_1\mathrm{ad}_{\gamma_i}}e^{-\epsilon_2\gamma_j}) \tag{19}$$

and expand in ϵ_1 and ϵ_2 we get the left hand side to be

$$e^{-i\epsilon_2\frac{J_j}{\hbar}}e^{-\frac{i}{\hbar}\epsilon_1\mathrm{ad}_{J_i}}e^{-i\epsilon_2\frac{J_j}{\hbar}} = e^{-i\epsilon_2\frac{J_j}{\hbar}}e^{-i\epsilon_2 e^{-\frac{i}{\hbar}\epsilon_1\mathrm{ad}_{J_i}}\frac{J_j}{\hbar}} \approx e^{-i\epsilon_2\frac{J_j}{\hbar}}e^{-i\epsilon_2\frac{J_j}{\hbar}-\frac{1}{\hbar^2}\epsilon_1\epsilon_2[J_i,J_j]} \approx 1 - \frac{1}{\hbar^2}\epsilon_1\epsilon_2[J_i,J_j] \tag{20}$$

Similarly, on the right hand side, we get

$$D(e^{\epsilon_2\gamma_j}e^{\epsilon_1\mathrm{ad}_{\gamma_i}}e^{-\epsilon_2\gamma_j}) \approx D(\mathbb{1} + \epsilon_1\epsilon_2[\gamma_i,\gamma_j]) = D(\mathbb{1} + \epsilon_1\epsilon_2\epsilon_{ijk}\gamma_k) \approx \mathbb{1} - \frac{i}{\hbar}\epsilon_1\epsilon_2\epsilon_{ijk}J_k \tag{21}$$

Equating the two leads to the angular momentum commutation relation

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \tag{22}$$

We see that unlike the different components of momentum which commute with each other, the different components of angular momentum do not commute. This is something we have already seen when studying the Stern-Gerlach device for spin $1/2$ where the devices rotated relative to each other measured incompatible observables.

Since the different components of angular momentum do not commute with each other, we cannot simultaneously diagonalize them. However, in addition to the three angular momentum components, we can also define the total angular momentum

$$\boldsymbol{J}^2 = \sum_i (J_i)^2 \tag{23}$$

It is easy to see that this operator commutes with all three angular momentum components J_i since $\mathrm{ad}_{\boldsymbol{J}^2} J_i = \sum_l \{J_l, \mathrm{ad}_{J_l} J_i\} = i \sum_{l,k} \epsilon_{lik} \{J_l, J_k\}$ which vanishes since the Levi-civita symbol is antisymmetric in l and k whereas the anticommutator is symmetric. This means that we can take $\boldsymbol{J}^2$ and one of the angular momentum components and use them to label the different angular momentum eigenstates. The convention is to choose J_z.

Let us consider a simultaneous eigenstate of $\boldsymbol{J}^2$ and J_z denoted by $|a, b\rangle$ such that

$$\boldsymbol{J}^2 |a, b\rangle = \hbar^2 a |a, b\rangle, \qquad J_z |a, b\rangle = \hbar b |a, b\rangle \tag{24}$$

Notice that since $\boldsymbol{J}^2$ is a sum of non-negative operators, $a \geq 0$. It is useful to define the non-hermitian 'ladder' operators

$$J_\pm = J_x \pm i J_y \tag{25}$$

which satisfy the commutation relations

$$[\boldsymbol{J}^2, J_\pm] = 0, \qquad [J_z, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_z \tag{26}$$

These relations should remind you of some commutation relations we discussed earlier. First, when we discussed translation operators, we had the relation

$$[\hat{x}, T_{\pm a}] = \pm a T_{\pm a} \tag{27}$$

The meaning of this relation is that the operator $T_{\pm a}$, acting on an eigenstate of $\hat{x}$, raises or lowers its eigenvalue by a. We also had similar relations for the harmonic oscillator algebra

$$[\hat{N}, a] = -a, \qquad [\hat{N}, a^\dagger] = a^\dagger \tag{28}$$

Here, again the interpretation is that $a^\dagger$ (a) increases (decreases) the eigenvalue of $\hat{N}$ by 1. The interpretation of the ladder operators $J_\pm$ is exactly the same. This means that

$$J_\pm |a, b\rangle = c_\pm |a, b \pm 1\rangle \tag{29}$$

Although the second equation in (26) is similar to the ladder operator algebra we encountered for translations and for the harmonic oscillator, the last equation is different and makes the properties of the angular momentum eigenstates quite distinct from these two other cases. To see this, let us recall what would be the analog of that relation in these two cases. For translation, the analog is $[T_a, T_{-a}] = 0$ since all translations commute. This does not impose any restrictions of the eigenvalues of the position operator (it is unbounded from above and below). For the harmonic oscillator, the anolagous relation is $[a, a^\dagger] = 1$. We saw that this relation had important implications on the spectrum of $\hat{N} = a^\dagger a$ which we found was bounded from below by 0 but unbounded from above. For the angular momentum algebra, we will see that the spectrum is bounded from both sides. First, we note that for any state $|u\rangle$

$$\langle u | \boldsymbol{J}^2 - J_z^2 | u \rangle = \langle u | J_x^2 + J_y^2 | u \rangle \geq 0 \tag{30}$$

This means that for a given fixed a, the eigenvalue of J_z^2, can never exceed a. In other words, there is a maximum value $b_{\max}$ such that

$$b \leq b_{\max} \leq \sqrt{a} \tag{31}$$

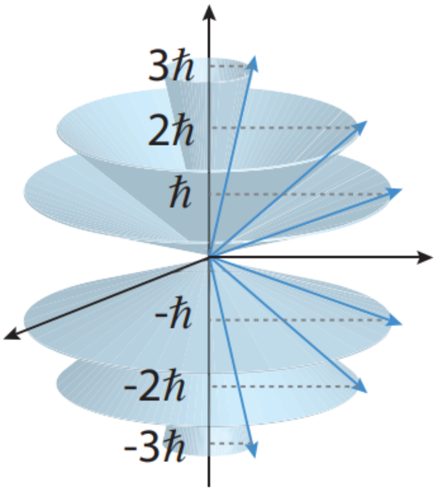

Figure 1: Illustration of the z component of angular momentum for $l = 3$. Note that the maximum value of J_z is smaller than the total angular momentum $\sqrt{12}\hbar \approx 3.46\hbar$.

Since we cannot raise the eigenvalue of J_z beyond b_{max}, J_+ has to annihilate $|a, b_{\text{max}}\rangle$. This allows us to write

$$\boldsymbol{J}^2|a, b_{\text{max}}\rangle = \hbar^2 a|a, b_{\text{max}}\rangle = [J_z^2 + \frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+]|a, b_{\text{max}}\rangle = \hbar^2 b_{\text{max}}[b_{\text{max}} + 1]|a, b_{\text{max}}\rangle \tag{32}$$

which implies

$$a = b_{\text{max}}[b_{\text{max}} + 1] \tag{33}$$

Now we can run the same argument for the minimum value $-\sqrt{a} \leq b_{\text{min}} \leq b$ to get

$$a = b_{\text{min}}[b_{\text{min}} - 1] \tag{34}$$

Comparing with (33), we see that $b_{\text{min}} = -b_{\text{max}}$. Now starting from b_{min}, we should be able to reach b_{max} by applying the raising operator a finite number of times. This yields

$$b_{\text{max}} - b_{\text{min}} = n, \qquad b_{\text{max}} = \frac{n}{2} \tag{35}$$

Thus, the eigenvalues of J_z are only allowed to be integer or half-integer. It is customary to use the notation where $b_{\text{max}} = j$ and $b = m$ and define

$$\boldsymbol{J}^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle, \qquad J^z|j, m\rangle = \hbar m|j, m\rangle \tag{36}$$

Pictorially, the components of angular momentum in quantum mechanics are quantized into $2j + 1$ equally spaced values whose maximum is **not** equal to the magnitude of the angular momentum as shown in Fig. 1. This counter-intuitive property follows from the last commutation relation in (26) which implies that $J_x^2 + J_y^2 = J_-J_+ + J_z$. Thus for the maximum value of $m = j$, $(J_x^2 + J_y^2)|j, j\rangle = \hbar j|j, j\rangle$ which means that the maximum possible value for any component of the angular momentum is always strictly smaller than the total angular momentum.

It is instructive to see what happens in the limit of large j. In this case, the total angular momentum corresponding to j is $\hbar\sqrt{j(j+1)}$. For large j, the spacing between different values of J_z remains the same $(= \hbar)$ but the size of the spacing relative to the total angular momentum, which we denote by Δ, decreases since $\Delta = \frac{1}{\sqrt{j(j+1)}}$. Furthermore, the ratio of the maximal value of J_z to the total angular momentum $\sqrt{\boldsymbol{J}^2}$ approaches 1 since $\frac{j(j+1)}{j^2} = 1 + \frac{1}{j} \approx 1$. This means that the limit of large j reproduces the expectation for a classical vector. We also note that when we did the path integral for spin, we had the factor s, which corresponds to our j here, multiplying the whole action. In the limit of large s, we can perform the saddle point approximation on the action and find that we recover the classical equations of motion for a classical unit vector.

Despite the unusual quantum mechanical nature of the angular momentum operators J_i, we still expect its components to transform like a vector under rotation. For instance, consider the rotated ket $|\alpha\rangle_R = e^{-\frac{i}{\hbar}\varphi J_z}|\alpha\rangle$. The expectation value of J_x in the rotated ket are related to their expectation value in the unrotated ket via

$$_R\langle\alpha|J_x|\alpha\rangle_R = \langle\alpha|e^{\frac{i}{\hbar}\varphi J_z}J_x e^{-\frac{i}{\hbar}\varphi J_z}|\alpha\rangle = \langle\alpha|e^{\frac{i}{\hbar}\varphi \mathrm{ad}_{J_z}}J_x|\alpha\rangle \tag{37}$$

Using the commutation relations (22), we find

$$\mathrm{ad}_{J_z}J_x = i\hbar J_y, \quad \mathrm{ad}_{J_z}^2 J_x = i\hbar \mathrm{ad}_{J_z}J_y = \hbar^2 J_x \tag{38}$$

which implies $\mathrm{ad}_{J_z}^{2n}J_x = \hbar^{2n}J_x$. Substituting in (37), we get

$$_R\langle\alpha|J_x|\alpha\rangle_R = \langle\alpha|J_x\sum_{n\text{ even}}\frac{(i\varphi)^n}{n!} + iJ_y\sum_{n\text{ odd}}\frac{(i\varphi)^n}{n!}|\alpha\rangle = \cos\varphi\langle\alpha|J_x|\alpha\rangle - \sin\varphi\langle\alpha|J_y|\alpha\rangle \tag{39}$$

A similar calculation for J_y yields

$$_R\langle\alpha|J_y|\alpha\rangle_R = \sin\varphi\langle\alpha|J_x|\alpha\rangle + \cos\varphi\langle\alpha|J_y|\alpha\rangle \tag{40}$$

More generally, the components of angular momentum transform as a vector under R

$$_R\langle\alpha|J_i|\alpha\rangle_R = \sum_l R_{il}\langle\alpha|J_l|\alpha\rangle \tag{41}$$

Although the expectation value of J_i behaves like a classical vector, when we consider the effect of J_i on kets, we will find a very surprising and counter intuitive result. Consider the case of spin-$\frac{1}{2}$ with $j = 1/2$. The Hilbert space is spanned by the two kets $|\pm\rangle$ satisfying

$$J_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle, \qquad J_+|+\rangle = J_-|-\rangle = 0 \tag{42}$$

If we introduce the vector notation

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{43}$$

we can write the J operators in terms of the Pauli matrices as $J_i = \frac{\hbar}{2}\sigma_i$. Let us now consider the action of rotation by angle φ around the z-axis on a general ket $|\alpha\rangle$

$$R_z(\varphi)|\alpha\rangle = e^{-\frac{i}{\hbar}\varphi J_z}(|+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle) = e^{-\frac{i}{2}\varphi}|+\rangle\langle+|\alpha\rangle + e^{\frac{i}{2}\varphi}|-\rangle\langle-|\alpha\rangle \tag{44}$$

The appearance of the factor of $\frac{1}{2}$ here has a very important implication: the action of rotation with $\varphi = 2\pi$ is non-trivial!

$$R_z(2\pi)|\alpha\rangle = -|\alpha\rangle \tag{45}$$

More generally, for angular momentum j we have

$$R_z(2\pi)|\alpha\rangle = e^{-\frac{2\pi i J_z}{\hbar}}\sum_{m=-j}^{j}|j,m\rangle\langle m,j|\alpha\rangle = \sum_{m=-j}^{j}e^{-2\pi i m}|j,m\rangle\langle m,j|\alpha\rangle = (-1)^{2j}|\alpha\rangle \tag{46}$$

Thus, for half-integer j, the action of 2π rotation gives a $-$ sign. Although the ket $-|\alpha\rangle$ represents the same state, we have seen in the discussion of Berry phase that such overall factor has physical consequences. For instance, if we have interference between two trajectories one involving the particle undergoing a 2π rotation and the other does not, we will get a destructive interference due to the extra minus sign if j is half-integer but a constructive interference if j is integer. This has implications for example when we consider electrical transport in solids where the momentum direction and spin are locked to each other. A trajectory where the momentum have rotated by 2π will be associated with an extra negative sign leading to some interesting quantum interference effects.