Lecture 18 & 19

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1 Addition of angular momenta

So far, we have discussed two types of angular momentum. There is intrinsic or spin angular momentum. This angular momentum can be thought of as a fundamental property of quantum objects that cannot be reduced to simpler properties. It satisfies the angular momentum algebra $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$ and its representations are generally labelled by non-negative half-integers s. We also discussed orbital angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ which is constructed from position and momentum operators and whose representations, the spherical harmonics, are labelled by non-negative integers l.

In general, a particle has both spin and orbital angular momentum and we will now discuss how to combine them together. This applies more generally to the addition of any pair of angular momentum operators as we will see. The first thing we want to consider when thinking of combining spin and orbital angular momenta is the structure of the Hilbert space. We have so far either focused on the spin component. For example, for spin $1/2$, we labelled the Hilbert space by the ket vectors $|+\rangle$ and $|-\rangle$. On the other hand, when considering the orbital or spatial dependence of the states, we ignored the spin part and expanded the kets in terms of eigenkets of the position operator $\hat{\boldsymbol{x}}$, $|\boldsymbol{x}\rangle$. However, since the position operator and the spin operator commute, we can construct basis labelled by both spin and position operators $|\pm, \boldsymbol{x}\rangle$ and write the wavefunction $\psi_\pm(\boldsymbol{x}) = \langle \pm, \boldsymbol{x}|\psi\rangle$ which can be written in the vector notation

$$\psi(\boldsymbol{x}) = \begin{pmatrix} \psi_+(\boldsymbol{x}) \\ \psi_-(\boldsymbol{x}) \end{pmatrix}, \qquad \int d^3\boldsymbol{x}\, \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}) = 1 = \int d^3\boldsymbol{x}\, [|\psi_+(\boldsymbol{x})|^2 + |\psi_-(\boldsymbol{x})|^2] \tag{1}$$

The basis $|\pm, \boldsymbol{x}\rangle$ can be understood as a tensor product $|\pm\rangle \otimes |\boldsymbol{x}\rangle$. A tensor product of two Hilbert spaces V_1 and V_2 with basis vectors $\boldsymbol{e}_l^{(1)}$, $l = 1, \dots, N_1$ and $\boldsymbol{e}_m^{(2)}$, $m = 1, \dots, N_2$ is the $N_1 \times N_2$-dimensional Hilbert space whose basis vectors can be constructed by combining the basis vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$.

The total angular momentum operator is the sum of spin and orbital angular momenta $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. Since $\boldsymbol{L}$ only acts on the orbital part and $\boldsymbol{S}$ only acts on the spin part, the proper way to write this as an operator acting on the full Hilbert space is

$$\boldsymbol{J} = \mathbb{1} \otimes \boldsymbol{L} + \boldsymbol{S} \otimes \mathbb{1} \tag{2}$$

This makes it clear that the operators $\boldsymbol{L}$ and $\boldsymbol{S}$ commute with each other. However, for simplicity, it is common to use the simpler but more sloppy notation where an operator $\mathbb{1} \otimes \mathcal{O}$ is just written as $\mathcal{O}$ with the understanding that the operator acts as the identity in the parts of the Hilbert space where its action is not defined. Since $[L_i, S_j] = 0$, the total angular momentum $\boldsymbol{J}$ satisfies the angular momentum algebra as expected.

Another example for the addition of angular momenta is if we take two spin $1/2$ particles. The Hilbert space for the two spins is 4-dimensional and is spanned by $|\sigma_1\rangle \otimes |\sigma_2\rangle$, $\sigma_{1,2} = \pm$. The total spin operator is given by

$$\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 = \boldsymbol{S} \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{S} \tag{3}$$

More generally, we would like to consider two angular momentum operators $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ and define the total angular momentum as $\boldsymbol{J} = \boldsymbol{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{J}_2$. The individual components of $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ commute with each other $[J_{1i}, J_{2j}] = 0$ which immediatley implies

$$[J_i, J_j] = [J_{1i} + J_{2i}, J_{1j} + J_{2j}] = [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}] = i\hbar\epsilon_{ijk}(J_{1k} + J_{2k}) = i\hbar\epsilon_{ijk}J_k \tag{4}$$

Thus, the total angular momentum satisfies the angular momentum commutation relations as expected. The action of a finite rotation by an angle φ around an axis $\hat{n}$ is given by

$$D[R_{\hat{n}}(\varphi)] = D_1[R_{\hat{n}}(\varphi)] \otimes D_2[R_{\hat{n}}(\varphi)] = e^{-i\varphi\frac{\boldsymbol{J}_1\cdot\hat{n}}{\hbar}} \otimes e^{-i\varphi\frac{\boldsymbol{J}_2\cdot\hat{n}}{\hbar}} \tag{5}$$

For infinitesmal φ, we can write $D[R_{\hat{n}}(\varphi)] = \mathbb{1} - i\frac{\varphi}{\hbar}\hat{n}\cdot[\boldsymbol{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{J}_2]$. This means that $\boldsymbol{J} = \boldsymbol{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{J}_2$ is the generator for a total rotation $D[R_{\hat{n}}(\varphi)]$ as expected. Our goal is to understand how the eigenfunctions of the total angular momentum are related to the eigenfunctions of the individual angular momentum operators.

Recall that, in general, we need to find a (maximal) set of commuting operators to label the basis of the Hilbert space. For the angular momentum algebra, we found that a natural choice of such commuting set of operators is given by the total angular momentum and any given component (taken to be the z-component by convention). However, for the case of addition of two angular momentum operators $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$, it turns out there are two natural choices. One is just to take $\boldsymbol{J}_1^2$, J_{1z}, $\boldsymbol{J}_2^2$ and J_{2z}. On the other hand, many physical contexts requires labelling the state with the total angular momentum, which is usually the quantity that couples to external physical probes, instead of individual components. In this case, it makes sense to use the total angular momentum $\boldsymbol{J}^2$ and its z-component J_z to label the states. Now since $\boldsymbol{J}^2 = \boldsymbol{J}_1^2 + \boldsymbol{J}_2^2 + 2\boldsymbol{J}_1\cdot\boldsymbol{J}_2 = \boldsymbol{J}_1^2 + \boldsymbol{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{2+}J_{1-}$. This means that $\boldsymbol{J}^2$ commutes with $\boldsymbol{J}_1^2$ and $\boldsymbol{J}_2^2$ but not J_{1z} or J_{2z}. Thus, the second possible choice for commuting operators is $\boldsymbol{J}^2$, J_z, $\boldsymbol{J}_1^2$ and $\boldsymbol{J}_2^2$. The eigenfunctions for the first choice are given by $|j_1 j_2; m_1 m_2\rangle$

$$\boldsymbol{J}_1^2|j_1j_2;m_1m_2\rangle = \hbar^2 j_1(j_1+1)|j_1j_2;m_1m_2\rangle, \qquad J_{1z}|j_1j_2;m_1m_2\rangle = \hbar m_1|j_1j_2;m_1m_2\rangle \tag{6}$$

$$\boldsymbol{J}_2^2|j_1j_2;m_1m_2\rangle = \hbar^2 j_2(j_2+1)|j_1j_2;m_1m_2\rangle, \qquad J_{2z}|j_1j_2;m_1m_2\rangle = \hbar m_2|j_1j_2;m_1m_2\rangle \tag{7}$$

The eigenfunctions for the second choice are given by $|j_1j_2;jm\rangle$

$$\boldsymbol{J}_1^2|j_1j_2;jm\rangle = \hbar^2 j_1(j_1+1)|j_1j_2;jm\rangle, \qquad \boldsymbol{J}_2^2|j_1j_2;jm\rangle = \hbar^2 j_2(j_2+1)|j_1j_2;jm\rangle \tag{8}$$

$$\boldsymbol{J}^2|j_1j_2;jm\rangle = \hbar^2 j(j+1)|j_1j_2;jm\rangle, \qquad J_z|j_1j_2;jm\rangle = \hbar m|j_1j_2;jm\rangle \tag{9}$$

1.1 The Clebsch-Gordan coefficients

Using the resolution of unity, we can expland one of these bases into the other

$$|j_1j_2;jm\rangle = \sum_{m_1,m_2} |j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle \tag{10}$$

The coefficients $\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle$ are known as the Clebsch-Gordan coefficients.

The Clebsch-Gordan coefficients have several important properties. First since $J_z - J_{1z} - J_{2z} = 0$, we can derive the relation

$$0 = \langle j_1j_2;m_1m_2|J_z - J_{1z} - J_{2z}|j_1j_2;jm\rangle = \hbar(m - m_1 - m_2)\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle \tag{11}$$

Thus, the Clebsch-Gordan coefficient $\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle$ vanishes unless $m = m_1 + m_2$. This makes sense since a state with total z-component of angular momentum m can only be decomposed into states whose z-component adds up to m.

A second relation is that

$$|j_1 - j_2| \le j \le j_1 + j_2 \tag{12}$$

This relation makes intuitive sense since the length of a sum of two vectors cannot exceed the sum of their lengths (which is realized if they are parallel) and cannot be smaller than the difference between their lengths (which is realized if they are anti-parallel). This relation implies that the dimension of the Hilbert space labelled by $|j_1 j_2; jm\rangle$ and $|j_1 j_2; m_1 m_2\rangle$ match. To see this, note that in the $|j_1 j_2; m_1 m_2\rangle$ basis, $-j_1 \leq m_1 \leq j_1$ and $-j_2 \leq m_2 \leq j_2$ leading to a total Hilbert space dimension of $(2j_1+1)(2j_2+1)$. On the other hand, counting the dimension of the $|j_1 j_2; jm\rangle$ gives $\sum_j (2j+1)$. Without loss of generality, we can assume $j_1 > j_2$ which implies that the sum over j goes from $j_1 - j_2$ to $j_1 + j_2$ yielding

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = (j_1+j_2)(j_1+j_2+1) - (j_1-j_2-1)(j_1-j_2) + 2j_2 + 1 = (2j_1+1)(2j_2+1) \quad (13)$$

Thus, the Clebsch-Gordan coefficients $\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle$ form a square matrix whose linear dimension is $(2j_1+1)(2j_2+1)$. It is relatively easy to see that this matrix is unitary since it relates to orthonormal basese to each other. As we will see later, the Clebsch-Gordan coefficients can also be chosen to be real which means that the matrix they form is an orthogonal matrix.

Before discussing the general procedure to construct the Clebsch-Gordan coefficients, it is useful to consider some simple examples. The first one is the addition of two spin $1/2$ particles. Here, $j_1 = j_2 = 1/2$ and we can drop the j_1 and j_2 indices from the basis kets for simplicitly. The first basis choice corresponds to labelling a state with the S_z component of each of the two particles yielding four states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$. The second basis is labelled by the total spin $j = s$ whose possible values are $s = 0, 1$. For $s = 0$, $m_s = 0$ whereas for $s = 1$, $m_s = 0, \pm 1$. These are called singlet ($s = 0$) and triplet ($s = 1$) representations. To expand the singlet state $|0,0\rangle$ in terms of the state $|\pm,\pm\rangle$, we use the fact that $m = 0 = m_1 + m_2$ which means that only $|+,-\rangle$ and $|-,+\rangle$ will contribute to the expansion. Thus,

$$|0,0\rangle = \alpha|+-\rangle + \beta|-+\rangle \quad (14)$$

Applying $J_+ = J_{1+} + J_{2+}$ to both sides, we get the condition $0 = (\alpha+\beta)|++\rangle$ which implies $\alpha = -\beta$. The normalization fixes the total magnitude of $|\alpha| = \frac{1}{\sqrt{2}}$ while the overall phase can be chosen arbitrarily. This gives the singlet state

$$|0,0\rangle = \frac{1}{\sqrt{2}}[|+-\rangle - |-+\rangle] \quad (15)$$

For the triplet states, we can perform a similar analysis. Due to the constraint, $m = m_1 + m_2$, the state $|1,1\rangle$ can only receive contribution from $|++\rangle$ which implies $|1,1\rangle = |++\rangle$. Similarly, $|1,-1\rangle$ only receives contribution from $|--\rangle$ which implies $|1,-1\rangle = |--\rangle$. Finally, $|1,0\rangle$ can be expanded in terms of $|+,-\rangle$ and $|-,+\rangle$. The requirement that this state is orthogonal to $|0,0\rangle$ immediately yields

$$|1,0\rangle = \frac{1}{\sqrt{2}}[|+-\rangle + |-+\rangle] \quad (16)$$

In summary

$$|0,0\rangle = \frac{1}{\sqrt{2}}[|+-\rangle - |-+\rangle], \quad |1,1\rangle = |++\rangle, \quad |1,-1\rangle = |--\rangle, \quad |1,0\rangle = \frac{1}{\sqrt{2}}[|+-\rangle + |-+\rangle] \quad (17)$$

In general, we can derive recursion relations for the Clebsch-Gordan coefficients by acting with the raising/lowering operators as follows

$$J_\pm |j_1 j_2; jm\rangle = (J_{1,\pm} + J_{2,\pm}) \sum_{m_1,m_2} |j_1 j_2; m_1 m_2\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \quad (18)$$

Using the relation

$$J_\pm |j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle \quad (19)$$

we get

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m+1)}|j_1 j_2; j, m \pm 1\rangle \\
&\qquad = \sum_{m_1,m_2} \sqrt{(j_1 \mp m_1)(j_1 \pm m_1+1)}|j_1 j_2; m_1 \pm 1, m_2\rangle\langle j_1 j_2; m_1 m_2|j_1 j_2; jm\rangle \\
&\qquad\qquad + \sum_{m_1,m_2} \sqrt{(j_2 \mp m_2)(j_2 \pm m_2+1)}|j_1 j_2; m_1, m_2 \pm 1\rangle\langle j_1 j_2; m_1 m_2|j_1 j_2; jm\rangle
\end{aligned} \tag{20}$$

Multiplying both sides by $\langle j_1, j_2; m_1, m_2|$ and using orhtonormality, yields

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m+1)}\langle j_1, j_2; m_1, m_2|j_1 j_2; j, m \pm 1\rangle \\
&\qquad = \sqrt{(j_1 \mp m_1+1)(j_1 \pm m_1)}\langle j_1 j_2; m_1 \mp 1 m_2|j_1 j_2; jm\rangle \\
&\qquad\qquad + \sqrt{(j_2 \mp m_2+1)(j_2 \pm m_2)}\langle j_1 j_2; m_1 m_2 \mp 1|j_1 j_2; jm\rangle
\end{aligned} \tag{21}$$

This relation allows us to determine all Clebsch-Gordan coefficients up to an overall phase that can be chosen by convention so that all coefficients are real. To do this, we start with the maximum possible values of $m_1 = j_1$ and $m_2 = j_2$. The only possible value of $m = j_1 + j_2$. This means that the space the CG coefficients for $m = j_1 + j_2$ correspond to a 1×1 matrix, whose single entry has to have absolute value 1 due to normalization. We can make the gauge choice such that it is real such that $\langle j_1 j_2; j_1 j_2|j_1, j_2; j, j_1+j_2\rangle = 1$. For the next largest $m = j_1 + j_2 - 1$, we have two possibilities, $(m_1, m_2) = (j_1, j_2 - 1)$ or $(m_1, m_2) = (j_1-1, j_2)$. The relation above can be used relate the two coefficients $\langle j_1 j_2; j_1, j_2-1|j_1, j_2; j, j_1+j_2-1\rangle$ and $\langle j_1 j_2; j_1 - 1, j_2|j_1, j_2; j, j_1 + j_2 - 1\rangle$ to the coefficient $\langle j_1 j_2; j_1 j_2|j_1, j_2; j, j_1 + j_2\rangle$ which we have already determined. Together with normalization, this gives two equations for the two unknowns $\langle j_1 j_2; j_1, j_2 - 1|j_1, j_2; j, j_1 + j_2 - 1\rangle$ and $\langle j_1 j_2; j_1 - 1, j_2|j_1, j_2; j, j_1 + j_2 - 1\rangle$ that can be used to fix them. This procedure can be repeated to obtain all CG coefficients.

2 Symmetries in Quantum Mechanics

We have already seen a few examples of symmetries in quantum mechanics. For instance, we have seen that the Schrödinger equation with a spherically symmetric potential commutes with the angular momentum operator. We have also seen that the free particle Schrödinger equation commutes with the translation operator which means that its eigenstates are also momentum eigenstates. These two are examples of continuous symmetries in quantum mechanics. We have also earlier discussed the parity operation when studying the harmonic oscillator and double well problems. Parity is an example of a discrete symmetry operation. In this and the following lectures, we will have a more systematic discussion of symmetries in quantum mechanics.

The connection between symmetries and conservation laws has already been well understood in classical mechanics. For instance, if the classical Hamiltonian of a system is independent of the position variable $\boldsymbol{x}$, i.e. the Hamiltonian is symmetric under translation $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{a}$, the Hamilton equations of motion imply $\dot{\boldsymbol{p}} = 0$ which means that $\boldsymbol{p}$ is a constant of motion. In the quantum theory, symmetries are described by unitary operators that leave the Hamiltonian invariant

$$\S\mathcal{H}\S^\dagger = \mathcal{H} \tag{22}$$

If $\S$ is a continuous symmetry labelled by some set of parameters λ^a, then we can define its infinitesimal generator G_a, as we have done for translation and rotation, via

$$\S(\lambda^a) = \mathbb{1} - \frac{i}{\hbar}\lambda^a G_a + O(\lambda^2) \tag{23}$$

where G_a are Hermitian operators $G_a^\dagger = G_a$.

The invariance of the Hamiltonian under the action of § implies

$$[G_a, \mathcal{H}] = 0 \tag{24}$$

That is, the infinitesimal symmetry generators commute with the Hamiltonian. An important consequence of this relation is that operators G_a are time independent in the Heisenberg picture since

$$i\hbar \frac{dG_a}{dt} = [G_a, \mathcal{H}] = 0 \tag{25}$$

This means that the operators G_a are constants of motion. This is the quantum version of the classical statement that a symmetry implies a conserved quantity or a constant of motion. In the Schrödinger picture where the time-dependent is in the states not the operators, the commutation relation (24) also has an important consequence. Consider an eigenket of a symmetry operator G at time $t = 0$

$$G|\lambda\rangle = \lambda|\lambda\rangle \tag{26}$$

Under time-evolution, $|\lambda\rangle$ will generally evolve to a different state (note that we are not assuming here that $|\lambda\rangle$ is a stationary state)

$$|\lambda, t_0; t\rangle = \mathcal{U}(t, t_0)|\lambda\rangle \tag{27}$$

Now recall that $\mathcal{U}(t, t_0)$ only depends on the Hamiltonian [1]. Thus, $[G_a, \mathcal{H}] = 0$ implies $[G_a, \mathcal{U}(t, t_0)] = 0$ which leads to

$$G|\lambda, t_0; t\rangle = G\mathcal{U}(t, t_0)|\lambda\rangle = \mathcal{U}(t, t_0)G|\lambda\rangle = \lambda\mathcal{U}(t, t_0)|\lambda\rangle = \lambda|\lambda, t_0; t\rangle \tag{28}$$

Thus, $|\lambda, t_0; t\rangle$ remains an eigenstate of G with eigenvalue λ at all times. Notice that this does not mean $|\lambda, t_0; t\rangle$ is a stationary state. For instance, let us consider a non-stationary state described by an even wavefunction $\psi(-x) = \psi(x)$ in the 1D harmonic oscillator. Such state will in general be described by a linear combination of even parity harmonic oscillator eigenstates. Under time evolution, the coefficients of such linear combination will change and the wavefunction itself will change but it will remain an even parity wavefunction. We will never introduce non-vanishing coefficients in the odd parity wavefunctions.

Symmetry has an important consequence for the spectrum of the Hamiltonian. Consider an eigenket of the Hamiltonian $|n\rangle$ with eigenvalue E_n, then the state $G|n\rangle$ is also an eigenket with the same eigenvalue since

$$\mathcal{H}G|n\rangle = G\mathcal{H}|n\rangle = E_n G|n\rangle \tag{29}$$

Then we have two possibilities:

1. $G|n\rangle = \lambda|n\rangle$ for some constant λ: this implies that $|n\rangle$ is also an eigenstate of G.
2. $G|n\rangle \neq \lambda|n\rangle$ for any λ: this means that $G|n\rangle$ and $|n\rangle$ represent different states i.e. the spectrum of the Hamiltonian $\mathcal{H}$ is degenerate.

We have already seen this with rotation symmetry which implies that the orbital angular momentum components commute with the Hamiltonian $[L_a, \mathcal{H}] = 0$. We have chosen to label the orbital angular momentum eigenspaces by the eigenvalues of $\boldsymbol{L}^2$ and L_z, denoted by $|l, m\rangle$. For the case of L_z, we have the first case above where $L_z|l, m\rangle \propto |l, m\rangle$. However for $L_{x,y}$, or more conveniently $L_\pm$, we have $L_\pm|l, m\rangle \propto |l, m \pm 1\rangle \neq \lambda|l, m\rangle$ for any λ. This represents the second case and implies that $|l, m\rangle$ and $|l, m \pm 1\rangle$ are degenerate energy eigenstates. This gives a $2l + 1$ degenerate space of eigenstates for a given l.

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[1] In the simplest cases it is just given by $e^{-\frac{i}{\hbar}\mathcal{H}(t-t_0)}$ but even in complicated cases where it is given by the Dyson series (Eq. 16 in Lecture 5), it only depends on $\mathcal{H}(t)$.

2.1 Parity symmetry

Parity or space-inversion is an operation that flips all spatial components sending a vector $\boldsymbol{x}$ to $-\boldsymbol{x}$. It is an example of the norm preserving operators we discussed in Lecture 13 which are described by an orthogonal matrix $RR^T = 1$ with $\det R = -1$ (it is not part of the special orthogonal group). Explicitly, the parity action on a 3D vector is described by the matrix

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{30}$$

Our goal now is to define a unitary operator π which represents the action of parity on kets in the same way $D(R)$ represented the action of rotation. The operator π should be distinguished from P in that it acts on kets that can be of arbitrary dimension. Similar to what we did with rotations, we require π to satisfy

$$\langle\alpha|\pi^\dagger x_i \pi|\alpha\rangle = \sum_j P_{ij}\langle\alpha|x_j|\alpha\rangle = -\langle\alpha|x_i|\alpha\rangle \tag{31}$$

Since $|\alpha\rangle$ is an arbitrary ket, this implies

$$\pi^\dagger \boldsymbol{x} \pi = -\boldsymbol{x}, \tag{32}$$

or equivalently, $\{\pi, \boldsymbol{x}\} = 0$.

The anticommutation of the parity operator and the position operator implies that the ket $\pi|\boldsymbol{x}\rangle$ satisfies

$$\boldsymbol{x}\pi|\boldsymbol{x}_0\rangle = -\pi\boldsymbol{x}|\boldsymbol{x}_0\rangle = -\boldsymbol{x}_0\pi|\boldsymbol{x}_0\rangle \tag{33}$$

Thus, the ket $\pi|\boldsymbol{x}_0\rangle$ is also a position eigenket with eigenvalue $-\boldsymbol{x}_0$ which means that $\pi|\boldsymbol{x}_0\rangle = e^{i\chi(\boldsymbol{x}_0)}|-\boldsymbol{x}_0\rangle$. The phase χ can be removed by an appropriate choice of gauge for the position eigenkets [2]. This implies that acting with π twice yields the same state $\pi^2|\boldsymbol{x}_0\rangle = |\boldsymbol{x}_0\rangle$, i.e. $\pi^2 = 1$. A unitary operator that squares to 1 is also hermitian since

$$\pi^\dagger = \pi^\dagger\pi^2 = (\pi^\dagger\pi)\pi = \pi \tag{34}$$

This also implies that the eigenvalues of the parity operator are ±1.

The action of parity on momentum can be understood as follows. In the position basis, the momentum is represented as $\boldsymbol{p} = -i\hbar\nabla_{\boldsymbol{x}}$ which suggests that $\boldsymbol{p}$ is also odd under parity. We can understand this from a more general basis-independent perspective by noting that translation and space inversion satisfy

$$Pt_{=}t_{-}P \tag{35}$$

which is the statement that translating a vector by then applying inversion is the same as translating the inverted vector by $-$. The same relation should apply for the operators T and π representing the action of translation and inversion on kets, respectively, i.e. $\pi T_{=}T_{-}\pi$. Taking to be infinitesimal and using the fact that momentum is the generator of translation $T_{=}\mathbb{1} - \frac{i}{\hbar} \cdot \boldsymbol{p}$ yields

$$\pi\boldsymbol{p} = -\boldsymbol{p}\pi \tag{36}$$

Thus, parity also anti-commutes with the momentum operator.

Finally, we want to understand the behavior of angular momentum under parity. As discussed previously, there is two types of angular momentum: (i) orbital angular momentum $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ and (ii) intrinsic angular momentum which is defined more abstractly as the generator of rotation. The fact that parity anticommutes with both $\boldsymbol{x}$ and $\boldsymbol{p}$ means that it commutes with the orbital angular momentum. For more general angular momenta that cannot be written in terms of $\boldsymbol{x}$ and $\boldsymbol{p}$, we need to use the definition of angular momentum as

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[2] For example, in 1D, we can define $|-x\rangle = \pi|x\rangle$ for $x > 0$.

generator of rotation and note that parity commutes with a general 3D rotation, $PR = RP$, which implies that it commutes with the infinitesimal generator of rotation

$$[\pi, \boldsymbol{J}] = 0 \tag{37}$$

This relation is a little strange. We generally expect a vector quantity to behave as a vector under rotation $V_i \mapsto R_{ij}V_j$ and to flip its sign under inversion $V_i \mapsto -V_i$. Although this is true for $\boldsymbol{x}$ and $\boldsymbol{p}$, it is not true for $\boldsymbol{J}$ which remains invariant under inversion. Such vector quantities which are invariant under inversion are called Axial or pseudovectors. We note that if we take the inner product of two vectors or pseudovectors, e.g. $\boldsymbol{x} \cdot \boldsymbol{p}$, the resulting quantity is a scalar that is invariant under both rotation and inversion. On the other hand, the inner product of a vector and a pseudovector yields a quantity that is odd under inversion. Such quantities are called pseudoscalars.

The behavior of the wavefunctions $\psi(\boldsymbol{x}) = \langle \boldsymbol{x}|\psi\rangle$ under inversion can be understood as follows. The wavefunction corresponding to the ket $\pi|\psi\rangle$ can be easily obtained as

$$\psi_\pi(\boldsymbol{x}) = \langle \boldsymbol{x}|\pi|\psi\rangle = \langle -\boldsymbol{x}|\psi\rangle = \psi(-\boldsymbol{x}) \tag{38}$$

If $|\psi_\pm\rangle$ is an parity eigenket with eigenvalue $\pm$, then we have

$$\langle \boldsymbol{x}|\pi|\psi_\pm\rangle = \pm\langle \boldsymbol{x}|\psi_\pm\rangle = \pm\psi_\pm(\boldsymbol{x}) \tag{39}$$

Equations (38) and (39) imply

$$\psi_\pm(-\boldsymbol{x}) = \pm\psi_\pm(\boldsymbol{x}) \tag{40}$$

This means that $+$ parity eigenvalues correspond to even wavefunctions and $-$ parity eigenvalues correspond to odd wavefunctions, as we discussed earlier.

If the Hamiltonian commutes with the parity operator $[\mathcal{H}, \pi] = 0$, then every non-degenerate eigenket of the Hamiltonian is also a parity eigenket. Degenerate eigenkets are generally not parity eigenvalues but we can choose particular linear combinations that are even or odd under parity. If $\mathcal{H}$ describes the motion of a particle in some potential $V(\boldsymbol{x})$, then the condition $[\mathcal{H}, \pi] = 0$ is equivalent to $V(-\boldsymbol{x}) = V(\boldsymbol{x})$. Let us now consider some simple cases of potentials satisfying this condition. For simplicity, we will restrict ourselves now to the 1D limit. The simplest case we can think of is that of a free particle where $V(x) = 0$. The eigenfunctions of the Hamiltonian are plane waves $\psi_k(x) = e^{ikx}$. These are not parity eigenfunctions since $\pi\psi_k(x) = e^{-ikx} = \psi_{-k}(x)$. This is consistent with the theorem above since the states $\psi_k(x)$ and $\psi_{-k}(x)$ are degenerate with the same energy $\frac{\hbar^2 k^2}{2m}$. Instead of the plane waves, we can choose to write the eigenfunctiosn of the Hamiltonian in terms of sin and cosine, $\psi_{\pm,k}(x) = e^{ikx} \pm e^{-ikx}$ which is proportional to $\cos kx$ for $+$ and $\sin kx$ for $-$ [3] Another example is the harmonic oscillator. Recall that the ground state of the 1D Harmonic oscillator was a Gaussian, which is even under parity. Now since the raising operator is a linear function of x and p, it is odd under parity

$$\{\pi, a\} = 0 = \{\pi, a^\dagger\} \tag{41}$$

This means that the n-th eigenstates of the harmonic oscillator $|n\rangle \propto (a^\dagger)^n|0\rangle$ has parity $(-1)^n$. This is consistent with the fact that the eigenstates of the 1D harmonic oscillator are bound states which are always non-degenerate, thus they have to be parity eigenstates.

One important physical consequence of parity symmetry is the following. Consider two parity eigenstates $|\sigma\rangle$ and $|\epsilon\rangle$ such that $\pi|\sigma\rangle = \sigma|\sigma\rangle$ and $\pi|\epsilon\rangle = \epsilon|\epsilon\rangle$, where σ, $\epsilon = \pm$. Now imagine an operator that commutes or anticommutes with parity

$$\pi^\dagger \hat{O}_\alpha \pi = \alpha \hat{O}_\alpha, \qquad \alpha = \pm \tag{42}$$

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[3] Note here that we should restrict ourselves to $k > 0$ since $\psi_{\pm,-k}(x) = \pm\psi_{\pm,k}(x)$. The special case of $k = 0$ has only the $+$ state since the $-$ state vanishes.

Then the matrix elements of such operator between the states $|\sigma\rangle$ and $|\epsilon\rangle$ vanish unless $\alpha\sigma\epsilon = 1$. To see this, consider

$$\langle\sigma|\hat{O}_\alpha|\epsilon\rangle = \langle\sigma|(\pi^\dagger)^2\hat{O}_\alpha\pi^2|\epsilon\rangle = \langle\sigma|\pi^\dagger(\pi^\dagger\hat{O}_\alpha\pi)\pi|\epsilon\rangle = \sigma\epsilon\alpha\langle\sigma|\hat{O}_\alpha|\epsilon\rangle \tag{43}$$

Thus, $\sigma\epsilon\alpha = -1$ implies $\langle\sigma|\hat{O}_\alpha|\epsilon\rangle = 0$.

This is an example of something called selection rules that plays an important role in understanding radiative transitions between different atomic states. A particular example is when the operator $\hat{O}$ is taken to be the position operator $\hat{\boldsymbol{x}}$. The selection rule above implies that only states with opposite parity can be connected by $\boldsymbol{x}$. An important consequence of this rule is that for a parity symmetric Hamiltonian with non-degenerate eigenstates $|n\rangle$, the dipole moment of any energy eigenstate, which is proportional to $\boldsymbol{x}$, vanishes since $\langle n|\boldsymbol{x}|n\rangle = 0$.

Next, let us discuss how electric and magnetic fields transform under parity. First, let us see what we expect classically. The equation for the classical Lorentz force is

$$\boldsymbol{F}_{\text{Lorentz}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{44}$$

Since $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$, we expect $\boldsymbol{F}$ to be odd under parity, i.e. to transform as a vector rather than a pseudovector. Since electric charge q is clearly a scalar, then the electric field should also be a vector, i.e. odd under parity. On the other hand, since the velocity $\boldsymbol{v}$ is odd under parity, $\boldsymbol{B}$ has to be even under parity i.e. $\boldsymbol{B}$ is a pseudovector. We can also verify that Maxwell equations are invariant under

$$\boldsymbol{x} \mapsto -\boldsymbol{x}, \qquad \boldsymbol{J} \mapsto -\boldsymbol{J}, \qquad \boldsymbol{E} \mapsto -\boldsymbol{E}, \qquad \boldsymbol{B} \mapsto \boldsymbol{B} \tag{45}$$

Thus, in the quantum theory, we expect the addition of an electromagnetic potential to a Hamiltonian to preserve parity i.e. $\pi\mathcal{H}(\boldsymbol{A},\varphi)\pi = \mathcal{H}(\boldsymbol{A},\varphi)$. This implies

$$\pi\mathcal{H}(\boldsymbol{A},\phi)\pi = \frac{1}{2m}(\boldsymbol{p} + q\pi\boldsymbol{A}\pi)^2 + q\pi\varphi\pi = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 + q\varphi \tag{46}$$

which implies

$$\pi\varphi\pi = \varphi, \qquad \pi\boldsymbol{A}\pi = -\boldsymbol{A} \tag{47}$$

up to a gauge transformation. This immediately gives

$$\pi\boldsymbol{E}\pi = \pi(-\nabla\varphi - \partial_t\boldsymbol{A})\pi = \nabla\varphi + \partial_t\boldsymbol{A} = -\boldsymbol{E}, \qquad \pi\boldsymbol{B}\pi = \pi(\boldsymbol{\nabla} \times \boldsymbol{A})\pi = \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B} \tag{48}$$

We will now discuss another very important symmetry in the quantum theory that turns out to be very subtle. It is called time-reversal symmetry.

2.2 Time-reversal symmetry

Time-reversal symmetry in the classical theory is the statement that, in the absence of dissipative forces, if $\boldsymbol{x}(t)$ is a solution to the classical equations of motion in a potential $V(\boldsymbol{x})$, then $\boldsymbol{x}(-t)$ is also a solution since the classical equation of motion $m\ddot{x} = -\boldsymbol{\nabla}V(\boldsymbol{x})$ is invariant under the replacement $t \mapsto -t$. Another way to say this is that if we have a movie for a particle or a collection of particles experiencing a potential and interacting with each other in the absence of dissipative forces, we cannot tell the difference between the movie playing forward or backward. In classical physics, we expect velocity to be odd under time reversal which implies that momentum and current are also odd. Again considering the expression for the Lorentz force, we can deduce that, under time-reversal, $\boldsymbol{E}$ is even while $\boldsymbol{B}$ is odd. We can explicitly verify that the Maxwell equations are invariant under

$$\boldsymbol{x} \mapsto \boldsymbol{x}, \qquad t \mapsto -t, \qquad \boldsymbol{J} \mapsto -\boldsymbol{J}, \qquad \boldsymbol{E} \mapsto \boldsymbol{E}, \qquad \boldsymbol{B} \mapsto -\boldsymbol{B} \tag{49}$$

The fact that $\boldsymbol{B}$ is odd under time-reversal makes intuitive sense since $\boldsymbol{B}$ is usually generated microscopically by some circulating currents which would switch direction under time-reversal. It is important to emphasize the following. If we study a system subject to a fixed external magnetic field, the system will appear to break time-reversal symmetry. This means that if we look at the time-reversed versions of trajectories of the system *keeping the external field fixed* we can tell the difference between a forward moving and a backward moving trajectory. Our goal now is to understand how time-reversal manifests in the quantum theory.

Let us start by considering the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(\boldsymbol{x},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x})\right)\psi(\boldsymbol{x},t) \tag{50}$$

Since this equation is linear rather than quadratic in time-derivatives, we see that for a solution $\psi(\boldsymbol{x},t)$, $\psi(\boldsymbol{x},-t)$ is not necessarily a solution. Does this mean time-reversal is broken in the quantum theory? This would be very surprising. One hint of how time-reversal should act in the quantum theory is the fact that time always appears in the quantum theory combined with a factor of i suggesting that the correct implementation of time-reversal symmetry should also involve complex conjugation. Indeed, if we can verify that whenever $\psi(\boldsymbol{x},t)$ is a solution to the Schrödinger equation, $\psi(\boldsymbol{x},-t)^*$ is also a solution.

But what does it mean to say that time-reversal action involves complex conjugation? In the operator formalism of the quantum theory, we have seen that symmetry transformations are implemented by unitary operators. However, the transformation that maps ψ to ψ^* is clearly non-unitary. The simplest way to see this is that $\langle\psi|\chi\rangle = \int d\boldsymbol{x}\psi^*(\boldsymbol{x})\chi(\boldsymbol{x}) \mapsto \int d\boldsymbol{x}\psi(\boldsymbol{x})\chi^*(\boldsymbol{x}) = \langle\psi|\chi\rangle^*$ i.e. the inner product of two kets is not invariant under complex conjugation. Does this mean that time-reversal is an invalid symmetry operation in the quantum theory?

The answer turns out to be no. We have just been using a restricted definition of symmetry operators. It is something we have already touched upon very briefly in one of the earlier lectures, but we can address more systematically now. The only measurable in the quantum theory is the absolute value of the overlap of two kets $|\langle\alpha|\beta\rangle|$ whose square gives the probability for the state $|\alpha\rangle$ to be measured in the state $|\beta\rangle$. This means that a symmetry operator $|\alpha\rangle \mapsto S|\alpha\rangle$ should satisfy the requirement

$$|\langle\alpha|\beta\rangle| \mapsto |\langle S\alpha|S\beta\rangle| = |\langle\alpha|\beta\rangle| \tag{51}$$

This is clearly satisfied by any unitary transformation $|\alpha\rangle \mapsto U|\alpha\rangle$ since $|\langle U\alpha|U\beta\rangle| = |\langle\alpha|U^\dagger U\beta\rangle| = |\langle\alpha|\beta\rangle|$. However, it is also satisfies with the complex conjugation operator $\psi \mapsto \psi^*$ defined above which maps $\langle\alpha|\beta\rangle$ to $|\langle\alpha|\beta\rangle^*$ which still preserves the absolute value of the overlap. Complex conjugation belongs more generally to a class of symmetry operators called anti-unitary. An anti-unitary operator is defined by $|\alpha\rangle \mapsto \Theta|\alpha\rangle$

$$\langle\Theta\alpha|\Theta\beta\rangle = \langle\alpha|\beta\rangle^*, \qquad \Theta(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1^*\Theta|\alpha\rangle + c_2^*\Theta|\beta\rangle \tag{52}$$

It is easy to see that the product of two anti-unitary operators is unitary which means that anti-unitary operators do not form a group by themselves. Instead, we can construct a group consisting of unitary and anti-unitary operators which contains unitary operators as a subgroup.

We have already seen an example of an anti-unitary operator which is the complex conjugation operator that we denote by $\mathcal{K}$. Consider for example the spin operators for a spin $1/2$ particle given by $S_i = \frac{\hbar}{2}\sigma_i$ where σ_i are the Pauli matrices given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{53}$$

We emphasize that these matrices are expressed in the basis $|z,\pm\rangle$. For example, $S_y = \frac{i\hbar}{2}(|z,-\rangle\langle z,+| - |z,+\rangle\langle z,-|)$. Since $\sigma_{x,z}$ are real while σ_y is imaginary, we have

$$S_{x,z} = S_{x,z}, \qquad S_y = -S_y \tag{54}$$

However, it is important to note that the representation of the complex conjugation operator is basis-dependent. For example, in the y-basis, $|y,\pm\rangle$, $S_y = \frac{\hbar}{2}(|y,+\rangle\langle y,+| - |y,-\rangle\langle y,-|)$. The action of complex conjugation in this basis is just $S_y = S_y$ which seems to contradict Eq. 54. To resolve this apparent contradiction, we note that like any other operator, is not basis-independent. Instead, under a change of basis, it changes as $\mapsto \sum_{ab} |a\rangle\langle a||b\rangle\langle b|$. For the example above, the transformation from the z basis to the y basis is implemented by a $\pi/2$ rotation around the x-axis, given by the unitary $U = e^{-\frac{i\pi}{2\hbar}S_x} = e^{-\frac{i\pi}{4}\sigma_x}$. Under this transformation, maps to $U^\dagger U = U^\dagger U^* = e^{\frac{i\pi}{2}\sigma_x} = i\sigma_x$. We can verify that the operator $i\sigma_x$ anticommutes with S_y expressed in the new basis. This example illustrates that a complex conjugation operator in one basis maps to a more general anti-unitary operator in a different basis.

So what is the most general anti-unitary operator? Since the product of any two anti-unitary operators is unitary, for any unitary operator, we can construct the combination Θ which is a unitary operator U. Noting that $^2 = 1$, we see that $\Theta = U$. Thus, any anti-unitary operator can be written as the product of a unitary operator times complex conjugation. We can now ask the question: are there any symmetry operators satisfying (51) that are not unitary or anti-unitary (up to unphysical phase factors)? The answer turns out to be no. A theorem by Wegner, whose proof is beyond the scope of this course, have shown that symmetries in the quantum theory can only be unitary or anti-unitary.

With our understanding of anti-unitary symmetries, we now revisit the question of time-reversal symmetry which as we argued before requires the introduction of complex conjugation. Time-reversal operator can be defined through its relation to the time evolution operator via

$$\mathcal{T}\mathcal{U}(t) = \mathcal{U}(-t)\mathcal{T} \tag{55}$$

This relation means that evolving a state by the time t then applying time-reversal is the same as applying time-reversal then evolving in the reverse time direction. Using the fact that the Hamiltonian is the generator of time-evolution $\mathcal{U}(dt) = \mathbb{1} - \frac{i}{\hbar}\mathcal{H}dt$, we find that

$$\mathcal{T}i\mathcal{H} = -i\mathcal{H}\mathcal{T} \tag{56}$$

Here, we kept the factor of i to account for the possibility that $\mathcal{T}$ is anti-unitary. According to Wegner theorem, we have two possibilities: either $\mathcal{T}$ is unitary which means that $\mathcal{T}i = i\mathcal{T}$ which implies $\mathcal{T}\mathcal{H} = -\mathcal{H}\mathcal{T}$. This equation implies that for any non-zero energy eigenvalue E of $\mathcal{H}$, there is a corresponding energy eigenvalues $-E$. This does not make sense physically which can be seen by considering simple examples such as the free particle Hamiltonian which we expect to be time-reversal symmetric but whose spectrum satisfies $E \geq 0$ and is unbounded from above. The other possibility is that $\mathcal{T}$ is anti-unitary

$$[\mathcal{T}, \mathcal{H}] = 0, \qquad \mathcal{T}^{-1}i\mathcal{T} = -i \tag{57}$$

The anti-unitarity of $\mathcal{T}$ has an important consequence. Physically, we expect that applying time-reversal twice to a state yields the same state. But recall that in the quantum theory, physical states correspond to rays in the Hilbert space rather than unique ket vectors. This means that for any ket $|\psi\rangle$, $\mathcal{T}^2|\psi\rangle = e^{i\theta}|\psi\rangle$ where θ is independent of $|\psi\rangle$ [4]. Thus, we can write $\mathcal{T} = e^{i\theta}\mathbb{1}$ which implies $U_\mathcal{T}U_\mathcal{T}^* = e^{i\theta}$ or equivalently $U_\mathcal{T} = e^{i\theta}U_\mathcal{T}^T$. Transposing both sides gives $U_\mathcal{T}^T = e^{i\theta}T_\mathcal{T} = e^{2i\theta}U_\mathcal{T}^T$ which implies $e^{2i\theta} = 1$. Thus, $\theta = 0$ or π which implies $\mathcal{T}^2 = \pm 1$. As we will see later, these two possibilities are realized for integer $(+)$ and half-integer $(-)$ spin.

The action of time-reversal on different physical observables can be understood based on simple physical considerations. First, we expect time-reversal to leave position eigenkets invariant $\mathcal{T}|x\rangle \propto |x\rangle$ which means that

$$\mathcal{T}\boldsymbol{x}\mathcal{T}^{-1} = \boldsymbol{x} \tag{58}$$

We also expect time-reversal to commute with the translation operator $T_{\boldsymbol{x}}$. Now since momentum is the generator of translation, we get

$$\mathcal{T}(i\boldsymbol{p})\mathcal{T}^{-1} = i\boldsymbol{p}, \qquad \Longrightarrow \qquad \mathcal{T}\boldsymbol{p}\mathcal{T}^{-1} = -\boldsymbol{p} \tag{59}$$

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[4] To see that this has to be the case, consider two different kets $|\psi_{1,2}\rangle$ and assume $\mathcal{T}^2|\psi_{1,2}\rangle = e^{i\theta_{1,2}}|\psi_{1,2}\rangle$. If $\theta_1 \neq \theta_2$, then the state $|\psi_1\rangle + |\psi_2\rangle$ will map to a different state $|\psi_1\rangle + e^{i(\theta_2-\theta_1)}|\psi_2\rangle$ under the action of $\mathcal{T}^2$.

Thus, momentum is odd under time-reversal as expected. This is what we would have obtained also from the position representation of the momentum operator $p_l = -i\hbar\frac{\partial}{\partial x_l}$. In fact, we can show a more general statement where any unitary operator must either commute with both $\boldsymbol{x}$ and $\boldsymbol{p}$ or anticommute with both $\boldsymbol{x}$ and $\boldsymbol{p}$ whereas an anti-unitary operator has to commute with one and anticommute with the other [5]. We can see this by acting with an arbitrary symmetry operator S on the Heisenberg commutation relation. Let us for simplicity consider the 1D case

$$S[x,p]S^{-1} = Si\hbar S^{-1} = [SxS^{-1}, SpS^{-1}] = SiS^{-1}\hbar \tag{60}$$

For unitary S, $SiS^{-1} = i$ which implies that S should commute with both x and p or anticommute with both x and p whereas for anti-unitary S, $SiS^{-1} = -i$ which implies that S should commute with one of x and p and anticommute with the other. An example of an operator that anticommute with both x and p is the parity operator π we studied last lecture. Time-reversal $\mathcal{T}$ is an example of an operator which commutes with x but anticommutes with p. Finally, we can consider the anti-unitary operator $\pi\mathcal{T}$ which anticommutes with x and commutes with p.

Finally, we expect time-reversal to commute with spatial rotations which implies

$$\mathcal{T}(iJ_a)\mathcal{T}^{-1} = iJ_a, \qquad \Longrightarrow \qquad \mathcal{T}J_a\mathcal{T}^{-1} = -J_a \tag{61}$$

This is again the result we expect at least for the case of orbital angular momentum $\boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{p}$. Note that the an operator that anticommutes with all angular momentum components have to to be antiunitary to respect the angular momentum commutation relations.

To understand the action of time-reversal symmetry on a system characterized by an arbitrary angular momentum labelled by a half-integer j, let us first consider the case of a spin 1/2 particle ($j = 1/2$). As we discussed earlier, the spin operators can be represented in terms of the Pauli matrices $S_i = \frac{\hbar}{2}\sigma_i$. Notice that the Pauli matrices σ_x and σ_z are real whereas the Pauli matrix σ_y is imaginary. This means that if we write $\mathcal{T} = \mathcal{U}_\mathcal{T}$ we have

$$\mathcal{U}_\mathcal{T}\sigma_x\mathcal{U}_\mathcal{T}^\dagger = -\sigma_x, \qquad \mathcal{U}_\mathcal{T}\sigma_y\mathcal{U}_\mathcal{T}^\dagger = \sigma_y, \qquad \mathcal{U}_\mathcal{T}\sigma_z\mathcal{U}_\mathcal{T}^\dagger = -\sigma_z \tag{62}$$

Thus, in the space defined by the three component vector $(\sigma_x, \sigma_y, \sigma_z)$, $\mathcal{U}_\mathcal{T}$ acts as a rotation around the y-axis by an angle π. Such rotation is represented by $\mathcal{U}_\mathcal{T} = e^{-\frac{i}{\hbar}\pi S_y} = e^{-\frac{i\pi}{2}\sigma_y} = i\sigma_y$. We now see that $\mathcal{T}^2 = \mathcal{U}_\mathcal{T}U_\mathcal{T}^* = e^{-i\pi\sigma_y} = -1$. Thus, the action of time-reversal on a spin 1/2 particle satisfies $\mathcal{T}^2 = -1$. We can anticipate the action on higher spins by recalling the formalism for addition of angular momentum we discussed in Lecture 16. There we found that combining an even number of spin $1/2$ particles can generally be expanded in terms of states whose total spin is integer while an odd number of spin $1/2$ particles can be expanded in terms of states whose total spin is half-integer. This suggests that the action of time-reversal on a state with n spin $1/2$ particles, defined via the tensor product $\mathcal{T}_n = \mathcal{T}\otimes\mathcal{T}\otimes\mathcal{T}\ldots$, satisfines $\mathcal{T}_n^2 = \mathcal{T}^2\otimes\mathcal{T}^2\otimes\mathcal{T}^2\cdots = (-1)^n$.

We can see this more generally by considering some arbitrary angular momentum algebra $[J_i, J_j] = i\epsilon_{ijk}\hbar J_k$. We can always choose two of the components to be real and one to be imaginary which is chosen to be J_y by convention. This means that

$$J_{x,z} = J_{x,z}, \qquad J_y = -J_y \tag{63}$$

This can be done explicitly by considering a specific representation $|j,m\rangle$. J_z is diagonal and real in this representation. $J_\pm$ can also be chosen to be real which means that $J_x = \frac{J_+ + J_-}{2}$ is real while $J_y = \frac{J_+ - J_-}{2i}$ is imaginary. Using Eq. 63 and the relation $\mathcal{T}J_a\mathcal{T}^{-1} = -J_a$ yields

$$\mathcal{U}_\mathcal{T}J_{x,z}\mathcal{U}_\mathcal{T}^\dagger = -J_{x,z}, \qquad \mathcal{U}_\mathcal{T}J_y\mathcal{U}_\mathcal{T}^\dagger = J_y, \tag{64}$$

which means that $\mathcal{U}_\mathcal{T} = e^{-\frac{i\pi}{\hbar}J_y}$ up to a phase. This gives $\mathcal{T}^2 = U_\mathcal{T}U_\mathcal{T}^* = e^{-\frac{2i\pi}{\hbar}J_y} = (-1)^{2j}$ where we used the fact that rotation by 2π around any axis gives $(-1)^{2j}$.

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[5] Here, we are not considering operators that mix $\boldsymbol{x}$ and $\boldsymbol{p}$