# Lecture 6

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### 0.1 Schrodinger and Heisenberg pictures

So far, we have taken the point of view that the operators, such as the spin, are time-independent whereas the states, represented by kets, change in time according to the time-evolution operator [1]. However, the states themselves are not directly observable in the quantum theory. The observables in the quantum theory are overlaps of diffent kets $\langle\alpha|\beta\rangle$, operator eigenvalues, and operator expectation values $\langle\alpha|A|\beta\rangle$. The first two are invariant under unitary transformations which means that they remain unchanged under unitary evolution. This leaves operator expectation values which encode the information about time-evolution through:

$$\langle\alpha,t|A|\beta,t\rangle = \langle\alpha|\mathcal{U}(t)^\dagger A\mathcal{U}(t)|\beta\rangle \tag{1}$$

Now notice that we will get the exact same result for operator expectation value if we assume that the operators evolve in time according to $A(t) = \mathcal{U}(t)^\dagger A\mathcal{U}(t)$ whereas the state kets are time-independent. This yields a completely equivalent formulation of the theory that gives the same physical results. The formulation we have discussed so far where states evolve in time is called the Schrödinger picture, whereas the formulation where operators change in time is called the Heisenberg picture. The two are related by

$$A^H(t) = \mathcal{U}^\dagger(t)A^S\mathcal{U}(t), \qquad |\alpha^S,t\rangle = \mathcal{U}(t)|\alpha^H\rangle \tag{2}$$

where we assume the operators and states in both pictures are equal at $t=0$.

The time-evolution for operators in the Heisenberg picture can be derived from Eq. **??** via

$$\begin{aligned}i\hbar\frac{d}{dt}A^H(t) &= i\hbar[\frac{d}{dt}\mathcal{U}^\dagger(t)]A^S\mathcal{U}(t) + \mathcal{U}^\dagger(t)A^S\frac{d}{dt}\mathcal{U}(t)] = [-\mathcal{U}^\dagger(t)\mathcal{H}^SA^S\mathcal{U}(t) + \mathcal{U}^\dagger(t)A^S\mathcal{H}^S\mathcal{U}(t)]\\ &= [A^H(t),\mathcal{H}^H(t)] = [A^H(t),\mathcal{H}]\end{aligned} \tag{3}$$

In the last equality, we assumed that the Hamiltonian in the Schrödinger picture does not depend explicitly on time so that $\mathcal{H}^H(t) = \mathcal{U}^\dagger(t)\mathcal{H}^S\mathcal{U}(t) = \mathcal{H}^S = \mathcal{H}$. The equation of motion for operators in the Heisenberg picture is called the Heisenberg equation of motion.

## 1 Harmonic oscillator

### 1.1 Creation and annihilation operators

We are now ready to apply this formalism to one of the most central problems in the quantum theory: the harmonic oscillator. In typical treatments of quantum mechanics, one introduces the Schrödinger equation for the wavefunction in position representation, then proceeds to find its solution for several specific potentials including the harmonic oscillators. We will present such treatment later. We will now instead present a different algebraic approach to the harmonic oscillator, originally due to Dirac, that does not rely on the

[1] Notice that this does not include explicit time-dependence of operators such as externally changing the Hamiltonian of a system

position basis and instead relies on the commutation relations. The Hamiltonian for the one-dimensional Harmonic oscillator is given by

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{4}$$

Here, $m$ is the mass of the particle and $\omega$ has units of inverse time or frequency. The crucial observation here is that the Hamiltonian is a sum of two squares $\hat{A}^2 + \hat{B}^2$ so we can use the identity $(\hat{A} + i\hat{B})(\hat{A} - i\hat{B}) = \hat{A}^2 + \hat{B}^2 + i[\hat{B}, \hat{A}]$ to express it in terms of the commutator $[\hat{A}, \hat{B}]$ and the **non-Hermitian** operators $\mathcal{D} = \hat{A} + i\hat{B}$ and $\mathcal{D}^\dagger = \hat{A} - i\hat{B}$. Identifying $\hat{A} = \frac{\hat{p}}{\sqrt{2m}}$ and $\hat{B} = \omega\hat{x}\sqrt{\frac{m}{2}}$, we can write

$$\mathcal{H} = -\frac{i}{2}\omega[\hat{x}, \hat{p}] + (\frac{\hat{p}}{\sqrt{2m}} + i\omega\hat{x}\sqrt{\frac{m}{2}})(\frac{\hat{p}}{\sqrt{2m}} - i\omega\hat{x}\sqrt{\frac{m}{2}}) = \hbar\omega\left[a^\dagger a + \frac{1}{2}\right] \tag{5}$$

where we defined the operators $a^\dagger$ and $a$, called the creation and annihilation operators, as

$$a = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}), \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p}) \tag{6}$$

The operators $a$ and $a^\dagger$ satisfy the simple commutation relations

$$[a, a^\dagger] = \frac{m\omega}{2\hbar}\frac{1}{m\omega}(-2i)[\hat{x}, \hat{p}] = \frac{1}{i\hbar}[\hat{x}, \hat{p}] = 1 \tag{7}$$

It is generally always a good idea to identify certain combinations of variables that appear together in a problem and identify what these might mean. In the definition of the creation and annihilation operators for the Harmonic oscillator, we see that $\hat{x}$ and $\frac{\hat{p}}{m\omega}$ has to have the same units. Recall that $\hat{p} = \hbar\hat{k}$ which means that $\hat{p}$ has units of action divided by length. This means that we can define a length scale $l^2 = \frac{\hbar}{m\omega}$. Then, we have

$$a = \frac{1}{\sqrt{2}l}(\hat{x} + i\frac{l^2}{\hbar}\hat{p}), \qquad a^\dagger = \frac{1}{\sqrt{2}l}(\hat{x} - i\frac{l^2}{\hbar}\hat{p}) \tag{8}$$

This also makes it clear that $a$ is dimensionless.

It is very useful to introduce the hermitian number operator $\hat{N} = a^\dagger a$. First, notice that the Hamiltonian has the simple form $\mathcal{H} = \hbar\omega[\hat{N} + \frac{1}{2}]$ which means that $[\mathcal{H}, \hat{N}] = 0$. Thus, we can construct the energy eigenstates by constructing the eigenstates of $\hat{N}$ which we will show has a simple form due to the simple commutation relation (7). First, note that for any state $|u\rangle$, the expectation value $\langle u|\hat{N}|u\rangle = \langle u|a^\dagger a|u\rangle = \|au\|^2$. This means that the eigenvalues of $\hat{N}$ are non-negative[2]. Thus, the energy expectation value satisfies $E = \langle u|\mathcal{H}|u\rangle \geq \frac{1}{2}\hbar\omega$.

Notice that the number operator $\hat{N}$ has simple commutation relations with the operators $a$ and $a^\dagger$. We can see this using the relation we proved in problem set one: $\mathrm{ad}_A BC := [A, BC] = (\mathrm{ad}_A B)C + B\mathrm{ad}_A C$:

$$[\hat{N}, a] = -\mathrm{ad}_a a^\dagger a = -a, \qquad [\hat{N}, a^\dagger] = -\mathrm{ad}_{a^\dagger} a^\dagger a = a^\dagger \tag{9}$$

These relations should remind you of something. Recall the commutation relations for the translation operator with the position operator

$$\hat{x}T_\alpha - T_\alpha\hat{x} = \alpha T_\alpha \tag{10}$$

This relation meant that $T_a$ acting on an eigenstate of $\hat{x}$ shifts its eigenvalue by $a$. To see how this relates to the commutation relations above, we identify $\hat{N}$ with the position operator. We then see that the first relation implies that, acting on an eigenstate of $\hat{N}$, $a$ lowers the eigenvalue by 1 whereas $a^\dagger$ raises its eigenvalue by 1. Explicitly, we write

$$\hat{N}|q\rangle = q|q\rangle, \qquad \langle q|q'\rangle = \delta_{q,q'} \tag{11}$$

where $q \geq 0$ since the operator $\hat{N}$ has non-negative eigenvalues as we discussed.

[2] This is true for any operator with the form $\hat{A} = \sum_i \mathcal{D}_i^\dagger \mathcal{D}_i$.

Acting with Eqs. 9 on $|q\rangle$, we get

$$\hat{N}(a|q\rangle) = (q-1)(a|q\rangle), \qquad \hat{N}(a^\dagger|q\rangle) = (q+1)(a^\dagger|q\rangle) \tag{12}$$

These relations mean that

$$a|q\rangle = C_q|q-1\rangle, \qquad a^\dagger|q\rangle = K_q|q+1\rangle \tag{13}$$

where $C_q$ and $K_q$ are constants. In the case of translations, we argued that we could choose the constant to be one since $T_a$ is unitary: $\langle x|T_a^\dagger T_a|x\rangle = \langle x|x\rangle$. This means we only needed to fix an overall phase that can be chosen to be 1. Here, the situation is different since $a$ is not a unitary operator $a^\dagger a \neq \mathbb{1}$. This implies that

$$|C_q|^2\langle q-1|q-1\rangle = \langle q|a^\dagger a|q\rangle = \langle q|\hat{N}|q\rangle = q\langle q|q\rangle, \qquad |C_q| = \sqrt{q} \tag{14}$$

where we used the normalization condition. Similarly, we find

$$|K_q|^2\langle q+1|q+1\rangle = \langle q|aa^\dagger|q\rangle = \langle q|(\hat{N}+1)|q\rangle = (q+1)\langle q|q\rangle, \qquad |K_q| = \sqrt{q+1} \tag{15}$$

Choosing the phase of $C_q$ and $K_q$ to be 1, we get

$$a|q\rangle = \sqrt{q}|q-1\rangle, \qquad a^\dagger|q\rangle = \sqrt{q+1}|q+1\rangle \tag{16}$$

So far, we only know that the spectrum of $\hat{N}$ is non-negative but we do not know whether it is continuous or discrete. To see that it should be the latter, we notice that we can iterate Eq. 16 to lower the eigenvalue $q$ by any integer

$$a^n|q\rangle = \sqrt{q(q-1)(q-2)\dots(q-n+1)}|q-n\rangle \tag{17}$$

If $q$ can take any value, then we can choose a large enough $n$ such that $q-n$ is negative contradicting the non-negativity of the spectrum we just established. The only resolution is that for $n \geq q$, the factor $q(q-1)(q-2)\dots(q-n+1)$ vanishes which means that $q$ has to be a (non-negative) integer. It also implies that $\hat{N}|0\rangle = a^\dagger a|0\rangle = 0$. Applying the raising operator $a^\dagger$, we see that the spectrum of $\hat{N}$ is given by non-negative integers $n \geq 0$

$$|n\rangle = \frac{a^\dagger}{\sqrt{n}}|n-1\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \qquad a|0\rangle = 0 \tag{18}$$

Thus, we can find the full spectrum of $\hat{N}$ once we know $|0\rangle$. The explicit form of $|0\rangle$ in the position representation can be obtained by writing

$$a|0\rangle = a\sum_x |x\rangle\langle x|0\rangle = \frac{1}{\sqrt{2}l}(\hat{x} + \frac{il^2}{\hbar}\hat{p})\sum_x |x\rangle\psi_0(x) = \frac{1}{\sqrt{2}l}\sum_x |x\rangle(x + l^2\partial_x)\psi_0(x) \tag{19}$$

To satisfy $a|0\rangle = 0$, we need

$$(x + l^2\partial_x)\psi_0(x) = 0 \tag{20}$$

which implies $\psi_0(x) = Ce^{-\frac{1}{2l^2}x^2}$ for some constant $C$ which is fixed by the normalization to be $C = \frac{1}{\pi^{1/4}\sqrt{l}}$.

### 1.2 Coherent states

We have some far discussed many operators. This includes hermitian operators such that the position $\hat{x}$, momentum $\hat{p}$, and Hamiltonian $\hat{H}$ operators. These operators have real spectrum and eigenstates which form a complete orthonormal basis. We have also encountered unitary operators such as translation and time-evolution which can be written as the exponentials of Hermitian generators and thus also have a complete set of orthonormal basis. How about the creation and annihilation operators? It turns out that their eigenstates have very interesting properties that differ significantly from what we discussed so far.

First, let us focus our attention to eigenkets. The eigenbras of $a$ ($a^\dagger$) are simply the duals of the eigenkets of $a^\dagger$ ($a$). The first thing to notice is that $a^\dagger$ has no eigenkets. We can see this by expanding a general ket into the $|n\rangle$ eigenbasis:

$$|u\rangle = \sum_{n=0}^{\infty} u_n |n\rangle \tag{21}$$

Now denote the smallest integer $n$ such that $u_n$ is non-zero by $N_{\min}$. This means that $\langle N_{\min}|u\rangle = u_{N_{\min}} \neq 0$. On the other hand,

$$\langle N_{\min}|a^\dagger|u\rangle = \sum_{n=N_{\min}}^{\infty} u_n \sqrt{n+1} \langle N_{\min}|n+1\rangle = 0 \tag{22}$$

Thus, it is impossible to satisfy the relation $a^\dagger|u\rangle = \lambda|u\rangle$ for $\lambda \neq 0$ i.e. $a^\dagger$ has no non-zero eigenkets. It is also impossible to have $a^\dagger|u\rangle = 0$ since this implies $0 = \langle u|aa^\dagger|u\rangle = \langle u|\hat{N}|u\rangle + 1$ which is impossible since $\langle u|\hat{N}|u\rangle \geq 0$. Visually, we can understand the absence of eigenkets of $a^\dagger$ by visualizing the states $|n\rangle$ as a semi-infinite lattice starting at zero and extending to $+\infty$ to the right. Acting with $a^\dagger$ shifts any state to the right by one site which can never leave any state invariant.

On the other hand, this picture suggests it is plausible that $a$ has some non-trivial eigenkets. To find these, we write

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha_n |n\rangle \tag{23}$$

Substituting in the eigenvalue equation $a|\alpha\rangle = \alpha|\alpha\rangle$, we get

$$a|\alpha\rangle = \sum_{n=1}^{\infty} \alpha_n \sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} \alpha_{n+1}\sqrt{n+1}|n\rangle = \alpha \sum_{n=0}^{\infty} \alpha_n |n\rangle \tag{24}$$

Taking the inner product of both sides with $\langle n|$, we get the relation

$$\alpha_{n+1} = \alpha \frac{\alpha_n}{\sqrt{n+1}} \tag{25}$$

which we can iterate to get

$$\alpha_n = \frac{\alpha^n}{\sqrt{n!}} \alpha_0 \tag{26}$$

Using $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$, we get

$$|\alpha\rangle = \alpha_0 \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!}|0\rangle = \alpha_0 e^{\alpha a^\dagger}|0\rangle = \alpha_0 e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle \tag{27}$$

In the last equality, we used the fact that $a|0\rangle = 0$ so $e^{\beta a}|0\rangle = |0\rangle$ for any $\beta$. The parameter $\alpha_0$ will be fixed by normalization as we will see later. Importantly, we see that the construction above is possible for any complex $\alpha$ which means that the spectrum of $a$ is complex and continuous. This may be puzzling given that the set $\{|\alpha\rangle\}$ should span the same space as the number eigenket basis $\{|n\rangle\}$ which is labelled by a discrete value. The resolution is that the coherent states $\{|\alpha\rangle\}$ are not orthogonal. The overlap of different coherent states will be discussed next lecture.