Lecture 7

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Last lecture, we discussed the algebraic solution of the Harmonic oscillator in terms of the creation and annihilation operators. We also discussed the coherent states which are eigenstates of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$. I would like to continue discussing the properties of coherent states

1 Coherent states: continued

Recall that the coherent states are eigenstates of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$ which can be written as

$$|\alpha\rangle = \alpha_0 e^{\alpha a^\dagger}|0\rangle \tag{1}$$

In the last lecture, we derived this expression from a series expansion. We will now show explicitly that the state $|\alpha\rangle$ is an eigenket of the annihilation operator by acting with a. To do this, we will introduce a set of identities that are very helpful when manipulating exponentials of operators. First, we have the Baker-Campbell-Hausdorff (BCH) formula

$$e^A B e^{-A} = e^{\mathrm{ad}A} B, \qquad e^{\mathrm{ad}A} = \sum_n \frac{1}{n!}\mathrm{ad}_A^n \tag{2}$$

The relation above also implies

$$e^{\mathrm{ad}_A} B^n = e^A B e^{-A} e^A B e^{-A} \cdots = (e^{\mathrm{ad}A} B)^n, \quad \implies \quad e^{\mathrm{ad}_A} f(B) = f(e^{\mathrm{ad}_A} B) \tag{3}$$

Acting with a on $|\alpha\rangle$ defined in (1), we get

$$a|\alpha\rangle = \alpha_0 a e^{\alpha a^\dagger}|0\rangle = \alpha_0 e^{\alpha a^\dagger} e^{-\alpha a^\dagger} a e^{\alpha a^\dagger}|0\rangle \tag{4}$$

Using Eq. 2, we can simplify $e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} = e^{-\alpha \mathrm{ad}_{a^\dagger}} a = \sum_n \frac{(-\alpha)^n}{n!}\mathrm{ad}_{a^\dagger} a = a - \alpha[a^\dagger, a] = a + \alpha$. Substituting in the expression above and noting that $a|0\rangle = 0$, we get

$$a|\alpha\rangle = \alpha_0 e^{\alpha a^\dagger}(a+\alpha)|0\rangle = \alpha|\alpha\rangle \tag{5}$$

To compute the normalization, we write

$$1 = \langle\alpha|\alpha\rangle = |\alpha_0|^2 \langle 0|e^{\alpha^* a} e^{\alpha a^\dagger}|0\rangle = |\alpha_0|^2 \langle 0|e^{\alpha^* a} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle \tag{6}$$

where we used the fact that inserting $e^{-\alpha^* a}$ next to the left of $|0\rangle$ does nothing since $a|0\rangle = 0$. To simplify the last equation, we use Eq. 3 to get

$$e^{\alpha^* a} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\alpha^* \mathrm{ad}_a} e^{\alpha a^\dagger} = e^{\alpha e^{\alpha^* \mathrm{ad}_a} a^\dagger} = e^{\alpha a^\dagger + |\alpha|^2 \mathrm{ad}_a a^\dagger} = e^{\alpha a^\dagger + |\alpha|^2} \tag{7}$$

Using the relation $\langle 0|e^{\alpha a^\dagger} = (e^{\alpha^* a}|0\rangle)^\dagger = (|0\rangle)^\dagger = \langle 0|$ and substituting in (6), we get

$$|\alpha_0|^2 e^{|\alpha|^2} = 1, \quad \implies \quad |\alpha_0| = e^{-\frac{1}{2}|\alpha|^2} \tag{8}$$

which gives the normalized coherent states

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle \tag{9}$$

We see that unlike the operator $\hat{N}$ whose eigenvalues were discrete and real, the eigenvalues of a, given α span all the complex numbers. Thus a has complex and continuous spectrum. Another property of the states $|\alpha\rangle$ that makes them very distinct from the eigenkets of hermitian operators is that they are not orthogonal. Instead, they satisfy

$$\begin{aligned}\langle\beta|\alpha\rangle &= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\langle 0|e^{\beta^* a} e^{\alpha a^\dagger}|0\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\langle 0|e^{\beta^* a} e^{\alpha a^\dagger} e^{-\beta^* a}|0\rangle \\ &= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha\beta^*)}\langle 0|e^{\alpha a^\dagger}|0\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha\beta^*)}\end{aligned} \tag{10}$$

In the last equality, we again used the relation $\langle 0| = (|0\rangle)^\dagger = (e^{\alpha^* a}|0\rangle)^\dagger = \langle 0|e^{\alpha a^\dagger}$. Although the coherent states are not orthogonal, they still form a complete basis in the sense that any state can be expanded in terms of them i.e. we can write a resolution of unity in terms of coherent states:

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \sum_{n,m}\frac{(a^\dagger)^n}{n!}|0\rangle\langle 0|\frac{(a)^m}{m!}\int d^2\alpha e^{-|\alpha|^2}\alpha^n(\alpha^*)^m \tag{11}$$

Here, we define the integration measure $d^2\alpha = d\alpha_r\, d\alpha_i$ where α_r and α_i are the real and imaginary parts of α. The integral over α can be evaluated by going to polar coordinates $\alpha = re^{i\phi}$

$$\int d^2\alpha e^{-|\alpha|^2}\alpha^n(\alpha^*)^m = \int_0^\infty r dr \int_0^{2\pi} d\phi r^{n+m} e^{i(n-m)\phi} e^{-r^2} = \pi n!\delta_{n,m} \tag{12}$$

Substituting in (11) gives

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \sum_n \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\langle 0|\frac{(a)^n}{\sqrt{n!}} = \sum_n |n\rangle\langle n| = \mathbb{1} \tag{13}$$

A basis that is complete (satisfies resolution of identity) but not orthogonal is called overcomplete. The coherent states play a crucial role in the development of path integrals as we will see later. They replace a *complete discrete* set of states describing the system by an *overcomplete continuous* set of states. Since they are labelled by a continuous parameter, they are the states that mostly resemble classical states and appear prominently in path integral approaches to quantum mechanics and semiclassical approximations as we will see later.

I would like to discuss two important properties of the coherent states. First, we can define a modified annihilation operator $a_\alpha = a - \alpha$ such that $a_\alpha|\alpha\rangle = 0$. This means that the coherent states are nothing but shifted versions of the ground state $|0\rangle$ of the

Another property of the coherent states is that they are minimum uncertainty states. This is compatible with their interpretation as the most classical states. In fact, below we will show a stronger statement: a state $|\alpha\rangle$ is a minimum uncertainty state in $\hat{x}$ and $\hat{p}$ if and only if there exists an a such that $a|\alpha\rangle = \alpha|\alpha\rangle$ or alternatively $a|\alpha\rangle = 0$. To prove this statement, we need to recall a few important steps in the proof for the uncertainty relation. The first step was writing $|u\rangle = (\hat{x} - \langle x\rangle)|w\rangle$ and $|v\rangle = (\hat{p} - \langle p\rangle)|w\rangle$ and using the Cauchy-Schwarz inequality

$$\|u\|^2\|v\|^2 \geq |\langle u|v\rangle|^2 \tag{14}$$

which is saturated if and only if $|u\rangle = \lambda|v\rangle$ for some λ [1]. This means that

$$\Delta x|w\rangle = \lambda\Delta p|w\rangle = \eta\frac{\tilde{l}^2}{\hbar}\Delta p|w\rangle \tag{15}$$

[1]To see this note that $\frac{1}{\|v\|^2}\|\|v\|^2|u\rangle - \langle u|v\rangle|v\rangle\|^2 = \|u\|^2\|v\|^2 - |\langle u|v\rangle|^2$

Here we have introduced the dimensionless parameter η which we can choose such that $|\eta| = 1$ and the parameter $\tilde{l}$ with units of length. Assuming the inequality is saturated, we have

$$|\langle\Delta x\rangle|^2|\langle\Delta p\rangle|^2 = |\langle\Delta x\Delta p\rangle|^2 = \frac{1}{4}|\langle[x,p]\rangle|^2 + \frac{1}{4}|\langle\{\Delta x,\Delta p\}\rangle|^2 \tag{16}$$

To satisfy the minimum uncertainty $|\langle\Delta x\rangle||\langle\Delta p\rangle| = \frac{\hbar}{2}$, we need $\langle\{\Delta x,\Delta p\}\rangle = 0$. Using Eq. 15, we find

$$0 = \langle\Delta x\Delta p + \Delta p\Delta x\rangle = \frac{2l^2}{\hbar}\mathrm{Re}[\eta]\langle(\Delta p)^2\rangle \tag{17}$$

which implies $\eta = \pm i$ [2]. Thus $a_\pm|u\rangle = \alpha|u\rangle$ where $\alpha = \frac{1}{\sqrt{2}l}(\langle x\rangle \pm i\frac{l^2}{\hbar}\langle p\rangle)$ and $a_\pm = \frac{1}{\sqrt{2}l}(\hat{x} \pm i\frac{l^2}{\hbar}\hat{p})$. However, only the $+$ solution is valid since $[a_\pm, a_\pm^\dagger] = \pm 1$ which means that a_+ is an annihilation operator whereas a_- is a creation operator which does not have any eigenfunctions.

1.1 Time dynamics of the harmonic oscillator

One final aspect we want to study for the Harmonic oscillator is the time dynamics. This illustrates the formalism we introduced last week. In particular, it demonstrates the usefulness of the Heisenberg picture which allows us directly to evaluate the time dependence of the variables x and p whose physical meaning is transpart instead of the less physically transparent eigenkets $|n\rangle$. The easiest way to evaluate the time dependence of $\hat{x}$ and $\hat{p}$ in the Heisenberg picture is to write them in terms of the creation and annihilation operators by inverting Eq. 3 from last lecture:

$$\hat{x} = \frac{l}{\sqrt{2}}(a + a^\dagger), \qquad \hat{p} = i\frac{\hbar}{\sqrt{2}l}(a^\dagger - a) \tag{18}$$

The Heisenberg equation of motion for a and $a^\dagger$ has a particularly simple form

$$\frac{da}{dt} = \frac{1}{i\hbar}[a,\mathcal{H}] = -i\omega[a,\hat{N}] = -i\omega a, \qquad \frac{da^\dagger}{dt} = \frac{1}{i\hbar}[a^\dagger,\mathcal{H}] = -i\omega[a^\dagger,\hat{N}] = i\omega a \tag{19}$$

The solution to these equations is

$$a(t) = a(0)e^{-i\omega t}, \qquad a(t)^\dagger = a(0)^\dagger e^{i\omega t} \tag{20}$$

Substituting in (18) yields

$$\hat{x}(t) = \frac{l}{\sqrt{2}}[(a(0) + a(0)^\dagger)\cos\omega t + i(a(0)^\dagger - a(0))\sin\omega t] = x(0)\cos\omega t + \frac{l^2}{\hbar}p(0)\sin\omega t \tag{21}$$

$$\hat{p}(t) = -\frac{\hbar}{l^2}x(0)\sin\omega t + p(0)\cos\omega t \tag{22}$$

These are the same as the equations for the classical phase space trajectory of the classical harmonic oscillator where the position and momentum variables oscillate out of phase.

It is instructive to evaluate $\hat{x}(t)$ directly by acting with the time evolution operator $x(t) = \mathcal{U}(t)^\dagger x(0)\mathcal{U}(t)$:

$$\hat{x}(t) = e^{\frac{i}{\hbar}\mathcal{H}t}\hat{x}e^{-\frac{i}{\hbar}\mathcal{H}t} \tag{23}$$

Thus, we can write

$$\hat{x}(t) = e^{\frac{i}{\hbar}t\,\mathrm{ad}_\mathcal{H}}\hat{x} \tag{24}$$

[2] This implication holds unless $|u\rangle$ is a momentum eigenstate. If that is the case, we can do the same argument replacing Δp with Δx

We now use

$$\mathrm{ad}_{\mathcal{H}}\hat{x} = \frac{1}{2m}\mathrm{ad}_{\hat{p}^2}\hat{x} = -i\frac{\hbar}{m}\hat{p} \tag{25}$$

$$\mathrm{ad}^2_{\mathcal{H}}\hat{x} = -i\frac{\hbar}{m}\mathrm{ad}_{\mathcal{H}}\hat{p} = -i\frac{\hbar\omega^2}{2}\mathrm{ad}_{\hat{x}^2}\hat{p} = \hbar^2\omega^2\hat{x} \tag{26}$$

From the second equation we see that $\mathrm{ad}^2_{\mathcal{H}}$ on $\hat{x}$ by multiplying by $(\hbar\omega)^2$. This means that

$$\mathrm{ad}^{2n}_{\mathcal{H}}\hat{x} = (\hbar\omega)^{2n}\,\hat{x}, \qquad \mathrm{ad}^{2n+1}_{\mathcal{H}}\hat{x} = (\hbar\omega)^{2n}\,\mathrm{ad}_{\mathcal{H}}\hat{x} = -i\frac{\hbar}{m}\,(\hbar\omega)^{2n}\,\hat{p} \tag{27}$$

Substituting in (24), we get

$$\begin{aligned}\hat{x}(t) &= \sum_n \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n \mathrm{ad}^n_{\mathcal{H}}\hat{x} = \sum_l \frac{1}{(2l)!}\left(\frac{it}{\hbar}\right)^{2l}\mathrm{ad}^{2l}_{\mathcal{H}}\hat{x} + \sum_l \frac{1}{(2l+1)!}\left(\frac{it}{\hbar}\right)^{2l+1}\mathrm{ad}^{2l+1}_{\mathcal{H}}\hat{x} \\ &= \hat{x}\sum_l \frac{(-1)^l}{(2l)!}\,(\omega t)^{2l} + \frac{1}{m\omega}\hat{p}\sum_l \frac{(-1)^l}{(2l+1)!}\,(\omega t)^{2l+1} = \hat{x}\cos\omega t + \frac{1}{m\omega}\hat{p}\sin\omega t\end{aligned} \tag{28}$$