Phys 251A Midterm

1. (30 points) True or false questions. No explanations required, just write "T" or "F".
   (a) A state for which the uncertainty $\langle(\Delta A)^2\rangle$ vanishes, where A is a Hermitian operator, must be an eigenstate of A.
   (b) The commutator of two Hermitian operators is Hermitian
   (c) For integers $N \geq 3$, there are $N \times N$ matrices X and P such that $[X, P] = i\hbar$.
   (d) When two Hermitian operators A and B commute, any eigenvector of A is also an eigenvector of B.
   (e) An operator that is both Unitary and Hermitian must square to the identity.
   (f) Suppose $\psi(x)$ is a bound state solution to the Schrodinger Wave equation. It is possible for the wavefunction and its derivative to vanish at the same point: $\psi(x_0) = 0$ and $\psi'(x_0) = 0$.
   (g) The Berry phase $\gamma = \int_{\boldsymbol{R}_i}^{\boldsymbol{R}_f} A \cdot d\boldsymbol{R}$ is gauge invariant, and only depends on the initial and final points and the path between them.
   (h) (three questions) Consider the Hamiltonian for a particle on a ring of radius R

   $$H = \frac{1}{2mR^2}\left(-i\hbar\partial_\phi - eA_\phi\right)^2$$

   which acts on wavefunctions $\psi(\phi)$ where $\psi(\phi + 2\pi) = \psi(0)$. Here, $A_\phi = \Phi/2\pi$ is the vector potential associated with a magnetic flux of Φ passing through the center of the ring that can vary continuously. Because A_ϕ is a constant, there is a gauge transformation that varies linearly in ϕ that can gauge it away leading to the gauge transformed Hamiltonian

   $$H' = -\frac{\hbar^2}{2mR^2}\partial_\phi^2$$

   Answer the following statements, true or false
      i. The wavefunctions and their boundary conditions are the same before and after the gauge transformation
      ii. The energy eigenvalues are the same before and after the gauge transformation
      iii. The spectrum of H is independent of Φ.

2. (30 points) Consider the three dimensional Harmonic oscillator

   $$H = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 + \frac{3}{2}) = \hbar\omega(N_1 + N_2 + N_3 + \frac{3}{2}) = \hbar\omega(N + \frac{3}{2})$$

   where a_1 and $a_1^\dagger$ are formed out of x and p_x and similarly for $a_{2,3}$ for y, z respectively. Consider the operators

   $$T_{ij} = a_i^\dagger a_j, \qquad i, j = 1, 2, 3$$

   (a) Write H and N_i in terms of certain T_{ij}
   (b) The commutator of two T operators is another T operator (they form a "closed algebra" under commutation). Confirm this by evaluating $[T_{ij}, T_{kl}]$. You may find the Kronecker delta symbol useful so that you can do all cases at once.
   (c) Calculate $[H, T_{ij}]$
   (d) Write the state $a_1^\dagger (a_3^\dagger)^2 |0\rangle$ in terms of a suitable T operator acting on the state $(a_3^\dagger)^3 |0\rangle$.

3. (40 points) *Squeezed States* Define the "squeezing operator"

   $$U(\lambda) = \exp(\lambda(aa - a^\dagger a^\dagger))$$

   where $[a, a^\dagger] = 1$ is the usual Harmonic oscillator commutation relation and

   $$x = \frac{l}{\sqrt{2}}(a + a^\dagger), \qquad p = \frac{i\hbar}{\sqrt{2}l}(a^\dagger - a)$$

   are the position and momentum operators in terms of the creation and annihilation operators and the oscillator length $l = \sqrt{\hbar/m\omega}$.

(a) For which values of λ is U unitary? For the rest of the problem, implicitly use these values of λ.

(b) Calculate $U^\dagger(\lambda)xU(\lambda)$ and $U^\dagger(\lambda)pU(\lambda)$, where x and p are the position and momentum operators. Hint: Consider differentiating with respect to λ and solving a differential equation, or use the adjoint power series.

(c) Starting with the ground state $|0\rangle$ of the Harmonic oscillator, which has uncertainties

$$\langle(\Delta\hat{x})^2\rangle_0 = l^2/2, \qquad \langle(\Delta\hat{p})^2\rangle_0 = \hbar^2/2l^2$$

Using the previous part, calculate the uncertainties $\langle(\Delta\hat{x})^2\rangle_\lambda$ and $\langle(\Delta\hat{p})^2\rangle_\lambda$, where the expectation values are taken in the state $|\lambda\rangle = U(\lambda)|0\rangle$. Show that the uncertainty relation is still saturated, and justify the terminology“squeeze operator”

(d) Now consider time evolving the squeezed state, such that $|\psi(t)\rangle = e^{-iHt/\hbar}|\lambda\rangle$, where $H = \hbar\omega(a^\dagger a + 1/2)$ is the Harmonic oscillator Hamiltonian. Evaluate the now time-dependent uncertainties $\langle(\Delta\hat{x})^2\rangle_{\psi(t)}$ and $\langle(\Delta\hat{p})^2\rangle_{\psi(t)}$ in terms of their values at $t = 0$.

Hint: One method is to use the Heisenberg picture, together with the results and methods you derived in previous parts of this problem. You may use the solution below to the Heisenberg equations of motion.

$$x(t) = x(0)\cos(\omega t) + \frac{l^2}{\hbar}p(0)\sin(\omega t), \qquad p(t) = p(0)\cos(\omega t) - \frac{\hbar}{l^2}x(0)\sin(\omega t),$$

As well as $\langle\{x,p\}\rangle_0 = 0$.

Squeezed states of photons are actively being applied to measure gravitational waves; appropriate squeezing can increase the signal to noise ratio, which is an inherent difficulty when the two kilometer detector length changes by a hundredths of the diameter of a proton while detecting black hole mergers.