# Phys 251A Midterm

1. (42 points) True or false questions. For each statement, determine whether it is true or false.

   **Briefly explain your reasoning.** It is sufficient to provide the key points; full derivations are not necessary.

   (a) The three-dimensional position and momentum operators $\boldsymbol{r} = (\hat{x}, \hat{y}, \hat{z})$ and $\boldsymbol{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ satisfy $\boldsymbol{r} \cdot \boldsymbol{p} - \boldsymbol{p} \cdot \boldsymbol{r} = 3i\hbar$

   (b) The knowledge of the time evolution operator $\mathcal{U}(t, t_0)$ is generally sufficient to construct the Hamiltonian of the system.

   (c) For a Schrodinger operator $\mathcal{O}^S = \mathcal{O}_1^S \mathcal{O}_2^S$, the corresponding Heisenberg operator is $\mathcal{O}^H(t)$ is equal to $\mathcal{O}_1^H(t)\mathcal{O}_2^H(t)$

   (d) The Heisenberg Hamiltonian and the Schrödinger Hamiltonian are equal if the Schrödinger Hamiltonians at different times commute.

   (e) For normalized kets $|u\rangle$ and $|v\rangle$, the operator $|u\rangle\langle v|$ is unitary.

   (f) It is possible to find an $N \times N$ matrix $a$ such that $[a, a^\dagger] = 1$.

   (g) The anticommutator of two Hermitian operators is Hermitian.

   (h) Position eigenstates satisfy $|-x\rangle = -|x\rangle$.

   (i) Consider the Harmonic oscillator Hamiltonian

   $$H_{\mathrm{HO}} = \hbar\omega(a^\dagger a + 1/2), \qquad [a, a^\dagger] = 1, \qquad \hat{N} = a^\dagger a \tag{1}$$

      i. The Heisenberg operator $\hat{N}(t)$ is time-independent

      ii. The Heisenberg operator $\hat{N}(t)$ for the Hamiltonian $H = H_{\mathrm{HO}} + \lambda(a^\dagger + a)^4$ is time-independent

      iii. The Heisenberg operator $\hat{N}(t)$ for the Hamiltonian $H = H_{\mathrm{HO}} + \lambda a^\dagger a^\dagger a a$ is time-independent

   (j) The Berry phase $\gamma = \oint \boldsymbol{A} \cdot d\boldsymbol{R}$ along any closed one-dimensional path is an integer multiple of $2\pi$.

   (k) In an Aharonov-Bohm experiment, a particle of charge $q$ experienced no phase difference between two paths which enclosed a flux of $\Phi$

      i. A particle with charge $2q$ will also experience no phase difference between the same paths

      ii. A particle with charge $q/2$ will experience a phase difference of $\pi$ between the same paths

2. (30 points) Consider the operator

   $$U(\lambda) = e^{\frac{i\lambda}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})} \tag{2}$$

   where $\lambda$ is a real number

   (a) Compute $\hat{x}_\lambda = U(\lambda)^\dagger \hat{x} U(\lambda)$

   (b) Compute $\hat{p}_\lambda = U(\lambda)^\dagger \hat{p} U(\lambda)$

   (c) Evaluate the commutator $[\hat{x}_\lambda, \hat{p}_\lambda]$

3. (28 points) Consider the Hamiltonian $H = \gamma B \boldsymbol{n} \cdot \boldsymbol{S}$, where $\boldsymbol{n}$ is a unit vector, and define $\omega = \gamma B$

   (a) For a state which points in the $+z$ at $t = 0$ subject to the field $\boldsymbol{n} = (1, 0, 0)$, find the state at time $t$. Interpretation: Describe the motion of the state over time. What classical motion does this quantum system's evolution correspond to?

   (b) For a state which points in the $+z$ at $t = 0$ subject to the field $\boldsymbol{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$, find the state at time $t$.

   (c) (For this section, you do **not** need to provide a proof. Simply state the result.) What direction of the time-independent field $\boldsymbol{n}$ is needed so that the spin goes from the $+z$ to $+x$ then $+y$ then back to $+z$ in a full period of oscillation?

**FORMULA SHEET**

1. Canonical Commutation Relation:

$$[\hat{x}, \hat{p}_x] = i\hbar$$

2. Time-dependent Schrödinger Equation:

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle = \mathcal{H}\,|\alpha, t_0; t\rangle$$

Time Evolution Operator:

$$|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0)|\alpha, t_0\rangle$$

so

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}(t, t_0) = \mathcal{H}(t)\mathcal{U}(t, t_0)$$

For Hamiltonian that is time-independent, the time-evolution operator:

$$\mathcal{U}(t) = e^{-\frac{i}{\hbar}\mathcal{H}t}$$

3. Heisenberg Picture and Schrödinger Picture

$$\hat{A}^H(t) = \mathcal{U}^\dagger(t)\hat{A}^S\mathcal{U}(t)$$

$$|\alpha^S, t\rangle = \mathcal{U}(t)\,|\alpha^H\rangle$$

4. Aharonov–Bohm Effect

A particle with charge $q$ traveling in a region with zero magnetic field but non-zero magnetic vector potential acquires a phase shift. Two particles with the same start and end points, but following different paths, will experience a phase difference proportional to the magnetic flux $\Phi$ through the area between the paths.:

$$\Delta\varphi = \frac{q\Phi}{\hbar}$$

5. Pauli Matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

6. Baker–Campbell–Hausdorff formula

$$e^A B e^{-A} = \sum_n \frac{1}{n!}\mathrm{ad}_A^n B = e^{\mathrm{ad}_A} B = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A, \cdots [A, [A,}_{\text{n times}} B]] \cdots ]]$$

$$\text{if } [A,B] \text{ commutes with both } A \text{ and } B\text{, then:} \qquad e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$