

Phys 251A Problem Set 2

1. Using that $|+z\rangle = |\uparrow\rangle$ and $|-z\rangle = |\downarrow\rangle$ are orthonormal, show that

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k, \quad \{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij}$$

where

$$S_z = \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|), \quad S_x = \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|), \quad S_y = \frac{\hbar}{2}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|).$$

We first compute the various products of spin operators. First we verify their squares.

$$\begin{aligned} S_z^2 &= \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) = \frac{\hbar^2}{4}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{\hbar^2}{4} \\ S_x^2 &= \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = \frac{\hbar^2}{4}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{\hbar^2}{4} \\ S_y^2 &= \frac{\hbar}{2}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) \frac{\hbar}{2}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) = \frac{\hbar^2}{4}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{\hbar^2}{4}. \end{aligned}$$

Next we calculate

$$\begin{aligned} S_x S_y &= \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) \frac{\hbar}{2}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) = i\frac{\hbar^2}{4}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}i\hbar S_z \\ S_y S_x &= \frac{\hbar}{2}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = -i\frac{\hbar^2}{4}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) = -\frac{1}{2}i\hbar S_z \\ S_y S_z &= \frac{\hbar}{2}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) = i\frac{\hbar^2}{4}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = \frac{1}{2}i\hbar S_x \\ S_z S_y &= \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \frac{\hbar}{2}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) = -i\frac{\hbar^2}{4}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = -\frac{1}{2}i\hbar S_x \\ S_z S_x &= \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = i\frac{\hbar^2}{4}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) = \frac{1}{2}i\hbar S_y \\ S_x S_z &= \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) = -i\frac{\hbar^2}{4}(-i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|) = -\frac{1}{2}i\hbar S_y. \end{aligned}$$

We see that $S_i S_j = -S_j S_i$ unless $i = j$, in which case $S_i^2 = \hbar^2/4$. This is summarized by the anticommutation relation

$$\{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij}$$

Similarly we see that $[S_i, S_j] = \pm i\hbar S_k$ as long as i and j are distinct, and where $k \neq i, j$. If i, j are the same then the commutator clearly vanishes. The \pm sign is determined by the cyclic ordering of i, j, k : it is positive if they are ordered as x, y, z or cyclic permutations of it such as z, y, x and negative if they are ordered as y, z, x and cyclic permutations of it.

Another object that has this property is the antisymmetric tensor ε_{ijk} , where $i, j, k \in \{1, 2, 3\}$ which is defined by its antisymmetry upon switching any two arguments together with $\varepsilon_{1,2,3} = 1$. Since cyclic permutations correspond to the combination of two transpositions, each of which flip the sign of ε_{ijk} , we see that ε_{ijk} is positive if i, j, k are cyclically ordered and negative if they are anti-cyclically ordered as claimed. We therefore conclude

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$$

2. Consider the spin operator $S_{\mathbf{n}} = \mathbf{n} \cdot \mathbf{S} = n_x S_x + n_y S_y + n_z S_z$ where $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit vector in an arbitrary direction characterized by a polar angle θ and azimuthal angle ϕ .
- (a) Construct the state $|\mathbf{n}+\rangle$, which is the eigenstate of $S_{\mathbf{n}}$ of positive eigenvalue, in terms of the $|\pm z\rangle$ basis states and the angles (θ, ϕ) .

In matrix form we have

$$S_{\mathbf{n}} = \frac{\hbar}{2} \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}.$$

We see that $\det S_{\mathbf{n}} = \frac{\hbar^2}{4} \lambda_1 \lambda_2 = -\frac{\hbar^2}{4} (n_x^2 + n_y^2 + n_z^2) = -\frac{\hbar^2}{4}$. Combining this with $\text{tr } S = \lambda_1 + \lambda_2$, we see that the eigenvalues are $\pm \hbar/2$. Solving for the eigenvalue with $+\hbar/2$ we have

$$S_{\mathbf{n}} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} u \\ v \end{pmatrix} \implies \cos(\theta)u + \sin \theta e^{-i\phi} v = u \implies u = \frac{\sin \theta e^{-i\phi} v}{1 - \cos \theta}.$$

A simpler answer can be obtained by using the double angle formulas $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ and $\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$. We then have

$$u = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-i\phi} v}{1 - \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-i\phi} v}{2 \sin^2 \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2} e^{-i\phi}}{\sin \frac{\theta}{2}} v$$

A normalized choice of u and v is then

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

- (b) Compute $\langle (\Delta S_x)^2 \rangle$ and $\langle (\Delta S_y)^2 \rangle$ in the state $|\mathbf{n}+\rangle$ in terms of the angles (θ, ϕ) . Convince yourself your answer makes sense by choosing some special (θ, ϕ) .

We have

$$\langle (\Delta S_x)^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} \left(1 - \left(\cos \frac{\theta}{2} \sin \frac{\theta}{2} (e^{i\phi} + e^{-i\phi}) \right)^2 \right) = \frac{\hbar^2}{4} (1 - \sin^2 \theta \cos^2 \phi)$$

and similarly

$$\langle (\Delta S_y)^2 \rangle = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} \left(1 - \left(\cos \frac{\theta}{2} \sin \frac{\theta}{2} (-ie^{i\phi} + ie^{-i\phi}) \right)^2 \right) = \frac{\hbar^2}{4} (1 - \sin^2 \theta \sin^2 \phi)$$

We see that the uncertainties are maximized if $\theta = 0, \pi$, i.e. the state is along the z axis. This makes sense, since $|\pm z\rangle$ are equal weight superpositions of $|\pm x\rangle$ and $|\pm y\rangle$. Similarly, the uncertainties are minimized, and zero, when $\theta = \pi/2$ and $\phi = 0, \pi$ for S_x and $\phi = \pm\pi/2$ for S_y . This makes sense because for these respective values the state is an eigenstate of S_x or S_y respectively.

- (c) Calculate $\langle [S_x, S_y] \rangle$ in the state $|\mathbf{n}+\rangle$. Show that the uncertainty relationship $\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle \geq \frac{1}{4} |\langle [S_x, S_y] \rangle|^2$ is satisfied.

Using that $[S_x, S_y] = i\hbar S_z$, and $\langle S_z \rangle_{\mathbf{n}} = \frac{\hbar}{2} (\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}) = \frac{\hbar}{2} \cos \theta$ we have

$$\frac{1}{4} |\langle [S_x, S_y] \rangle|^2 = \frac{1}{4} \left(\frac{\hbar}{2} \cos \theta \right)^2 = \frac{\hbar^2}{16} \cos^2 \theta$$

The claimed uncertainty relation is then equivalent to the inequality

$$(1 - \sin^2 \theta \cos^2 \phi)(1 - \sin^2 \theta \sin^2 \phi) \geq \cos^2 \theta$$

To show this inequality we move everything to the left side and show that the result is positive:

$$\begin{aligned} (1 - \sin^2 \theta \cos^2 \phi)(1 - \sin^2 \theta \sin^2 \phi) - \cos^2 \theta &= 1 - \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) - \cos^2 \theta + \sin^4 \theta \cos^2 \phi \sin^2 \phi \\ &= \sin^4 \theta \cos^2 \phi \sin^2 \phi \geq 0 \end{aligned}$$

- (d) For which \mathbf{n} is the uncertainty product $\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle$ maximized?

The uncertainty product

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = \frac{\hbar^2}{16} (1 - \sin^2 \theta \cos^2 \phi)(1 - \sin^2 \theta \sin^2 \phi) = \frac{\hbar^2}{16} (1 - n_x^2)(1 - n_y^2)$$

is clearly always less than 1. It achieves its maximal value of 1 at $\theta = 0, \pi$, irrespective of the value of ϕ , which corresponds to states polarized along the z -axis.

3. Consider a momentum operator p that has eigenstates $|p\rangle$. Define an infinitesimal boost operator by the relationship $B(dp)|p_0\rangle = |p_0 + dp\rangle$. Write $B(dp) = 1 + iWdp$

- (a) Show that $B(dp)$ is unitary

We can write

$$B(dp) = \int \frac{dp_0}{2\pi} |p_0 + dp\rangle \langle p_0|$$

where momentum states are normalized as $\langle p|p'\rangle = 2\pi\delta(p - p')$. Taking the conjugate we have

$$B(dp)^\dagger = \int \frac{dp}{2\pi} (|p_0 + dp\rangle \langle p_0|)^\dagger = \int \frac{dp}{2\pi} |p_0\rangle \langle p_0 + dp|$$

which clearly has the inverse action of taking $|p_0 + dp\rangle \rightarrow |p_0\rangle$, for any p_0 . Thus $B(dp)^\dagger = B(dp)^{-1}$ and $B(dp)$ is unitary. Note that we can similarly conclude that $B(dp)^\dagger = B(-dp)$ by shifting the integration variable $p_0 \rightarrow p_0 - dp$.

- (b) Write $B(dp) = 1 + iWdp$. Show that W is Hermitian.

Using that $B(dp)^\dagger = B(-dp)$ we have

$$B(dp)^\dagger = 1 - iW^\dagger dp = B(-dp) = 1 - iWdp$$

so that we can conclude $W = W^\dagger$ by comparing the second and fourth expressions.

- (c) Calculate $[W, p]$, and write an ansatz of W in terms of x that satisfies this commutation relationship and thus generates the boost $B(dp)$.

Let us first calculate $[p, B(dp)]$. Acting on a ket $|p_0\rangle$ we obtain

$$[p, B(dp)]|p_0\rangle = (p_0 + dp - p_0)|p_0 + dp\rangle = dp|p_0 + dp\rangle = dpB(dp)|p_0\rangle.$$

and we can therefore conclude $[p, B(dp)] = dpB(dp)$ since operators are determined by their action on

basis vectors. Substituting $B(dp) = 1 + iWdp$ and neglecting dp^2 we have

$$[p, B(dp)] = idp[p, W] = dpB(dp) = dp$$

so that $[p, W] = -i$. We therefore write $W = x/\hbar$, to match the canonical commutation relation.

4. Using the canonical commutation relationship $[x, p] = i\hbar$ and the identity proved in the last problem set, $\text{ad}_A(BC) = \text{ad}_A(B)C + B \text{ad}_A(C)$ where $\text{ad}_X(Y) = [X, Y]$,

- (a) Show that $[p, x^n] = -i\hbar nx^{n-1}$ (hint: use induction)

The base case, $n = 1$ is $[p, x] = -i\hbar$. Assuming the result for $n - 1$, we have, using the “product rule”

$$[p, x^n] = x[p, x^{n-1}] + x^{n-1}[p, x] = x(-i\hbar(n-1)x^{n-2}) + x^{n-1}(-i\hbar) = -i\hbar nx^{n-1} \quad (1)$$

as desired

- (b) Show that $[x, f(p)] = i\hbar f'(p)$ for any function f that can be expanded in a power series

The same reasoning as above shows that $[x, p^n] = i\hbar np^{n-1}$. By linearity of the commutator we then have

$$[x, f(p)] = \sum_n f_n [x, p^n] = i\hbar \sum_n n f_n p^{n-1} = i\hbar f'(p)$$

- (c) Evaluate $[x^2, p^2]$

Using the commutator identity and part (a) we have

$$[x^2, p^2] = [x^2, p]p + p[x^2, p] = (2i\hbar x)p + p(2i\hbar x) = 2i\hbar\{x, p\}.$$

- (d) Show that $e^{ipa/\hbar} |x_0\rangle$ is an eigenvector of x and find the corresponding eigenvalue.

Acting with the position operator x and using part (b) we have

$$xe^{ipa/\hbar} |x_0\rangle = (e^{ipa/\hbar} x + [x, e^{ipa/\hbar}]) |x_0\rangle = (e^{ipa/\hbar} x + i\hbar(ia/\hbar)e^{ipa/\hbar}) |x_0\rangle = (x_0 - a) |x_0\rangle$$

5. Consider a Gaussian wavepacket $\psi(x) = ce^{\frac{-x^2}{2\sigma^2}}$, where c is an arbitrary constant that could be chosen so that the state is normalized. Calculate $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ in the quantum state described by the wavepacket. Show that the uncertainty relation $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4}$ is saturated.

We note that this wavepacket is symmetric under $x \rightarrow -x$ so $\langle x \rangle = \langle p \rangle = 0$ (both integrals are odd under $x \rightarrow -x$. We then have

$$\langle(\Delta x)^2\rangle = \langle x^2 \rangle = \frac{\int x^2 e^{-x^2/\sigma^2} dx}{\int e^{-x^2/\sigma^2} dx} = \frac{\frac{d}{d\sigma^{-2}} \int e^{-x^2/\sigma^2} dx}{\int e^{-x^2/\sigma^2} dx} = \frac{\frac{d}{d\sigma^{-2}} \sqrt{\pi/\sigma^{-2}}}{\sqrt{\pi/\sigma^{-2}}} = \frac{1}{2}\sigma^2$$

and

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle = \frac{\langle p\psi|p\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int | -i\hbar\partial_x e^{-x^2/(2\sigma^2)} |^2 dx}{\int e^{-x^2/\sigma^2} dx} = \frac{\hbar^2 \int x^2 e^{-x^2/\sigma^2} dx}{\sigma^4 \int e^{-x^2/\sigma^2} dx} = \frac{\hbar^2}{\sigma^4} \langle x \rangle^2 = \frac{\hbar^2}{2} \sigma^{-2}$$

such that the uncertainty product is exactly $\hbar^2/4$, and Heisenberg's uncertainty inequality is saturated.