Phys 251A Problem Set 3
Date posted: September 22, 2023
Due date: September 28, 2023

1. Solve the spin procession problem in the Heisenberg picture. The Hamiltonian is

$$H = \omega S_z$$

where ω is proportional to the magnetic field.

(a) Write the Heisenberg equations of motion for $S_x(t)$ and $S_y(t)$ and solve them.

The Heisenberg equations are

$$-i\hbar \frac{d}{dt} S_x(t) = [H, S_x] = \omega [S_z, S_x] = i\hbar\omega S_y$$

and

$$-i\hbar \frac{d}{dt} S_y(t) = [H, S_y] = \omega [S_z, S_y] = -i\hbar\omega S_x,$$

such that

$$\frac{d}{dt} S_x(t) = -\omega S_y(t), \qquad \frac{d}{dt} S_y(t) = \omega S_x(t).$$

The above equations have the solution

$$S_x(t) = S_x(0)\cos(\omega t) - S_y(0)\sin(\omega t), \qquad S_y(t) = S_y(0)\cos(\omega t) + S_x(0)\sin(\omega t),$$

such that

$$S_x(t) = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}, \qquad S_y(t) = \frac{\hbar}{2}\begin{pmatrix} 0 & -ie^{i\omega t} \\ ie^{-i\omega t} & 0 \end{pmatrix}.$$

Here we have substituted the typical Schrodinger picture operators, in the z-basis, for the Heisenberg operators at $t = 0$.

(b) For a state in the $|\boldsymbol{n}+\rangle$ direction find the probability to measure $S_x = +\hbar/2$ at time t.

The measurement is with respect to the $+\hbar/2$ eigenstate of $S_x(t)$, which we denote $|x(t), +\rangle$, at time t. Using the previous part we find that

$$|x(t), +\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\omega t} \end{pmatrix}$$

has eigenvalue $+\hbar/2$ under $S_x(t)$. From the previous problem sets we have $|\boldsymbol{n}+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}^T$ such that the probability to measure $+\hbar/2$ for S_x is

$$\begin{aligned} |\langle x(t), +|\boldsymbol{n}+\rangle|^2 &= \frac{1}{2}\left|\cos(\theta/2) + \sin(\theta/2)e^{i(\phi+\omega t)}\right|^2 \\ &= \frac{1}{2}(\cos^2(\theta/2) + \sin^2(\theta/2) + 2\cos\theta/2\sin\theta/2\cos(\phi+\omega t)) = \frac{1}{2}(1 + \sin(\theta)\cos(\phi+\omega t)) \end{aligned}$$

(c) Find the expectation value of S_x at time t in the state $|\boldsymbol{n}+\rangle$.

Using $|\boldsymbol{n}+\rangle = \cos(\theta/2)\,|\uparrow\rangle + e^{i\phi}\sin(\theta/2)\,|\downarrow\rangle$ from the last problem set, we can calulate at $t = 0$

$$\langle S_x(0)\rangle = \frac{\hbar}{2}\cos(\theta/2)\sin(\theta/2)(e^{i\phi} + e^{-i\phi}) = \frac{\hbar}{2}\sin(\theta)\cos(\phi) = \frac{\hbar}{2}n_x$$

so that at other times

$$\langle S_x(t)\rangle = \langle S_x(0)\rangle\cos(\omega t) - \langle S_y(0)\rangle\sin(\omega t) = \frac{\hbar}{2}\left(n_x\cos(\omega t) - n_y\sin(\omega t)\right) = \frac{\hbar}{2}\sin(\theta)\cos(\omega t + \phi)$$

(d) Convince yourself that your answers to the two parts above make sense for various limiting values of $|\boldsymbol{n}\rangle$ (e.g. along the z axis or in the x-y plane).

Along the z axis we have $n_x = n_y = 0$ and $\theta = 0, \pi$ such that the probability to measure S_x is always $1/2$ and $\langle S_x(t)\rangle = 0$. This makes sense; the $|\pm z\rangle$ states are eigenstates of H and are therefore time independent. More physically, the spin is precessing around the z axis and so states that are polarized along z do not see this precession; the z-component of the spin is time-independent.

If $\theta = \pi/2$ such that the spin is precessing in the x-y plane, the probability to measure $\hbar/2$ for S_x reaches 1 at certain times, and the expectation value of $\langle S_x\rangle$ can also reach $\hbar/2$. This also makes sense, since for these times the spin points exactly along the x direction.

2. Suppose that your TF made an error and wrote the Hamiltonian

$$H = H_{11}\,|1\rangle\langle 1| + H_{22}\,|2\rangle\langle 2| + H_{12}\,|1\rangle\langle 2|$$

where $|1\rangle$ and $|2\rangle$ are orthonormal. Simplify the time evolution operator and describe its action (hint: use power series, and you can assume $H_{11} = H_{22} = 0$ for simplicity). What fundamental principle of quantum mechanics is now violated, and why do its predictions no longer make sense?

The resulting Hamiltonian, with the simplifying assumption, is simply $H = H_{12}\,|1\rangle\langle 2|$. Note that $H^2 = 0$, so that the time evolution operator is

$$U = e^{-iHt/\hbar} = 1 - \frac{i}{\hbar}tH_{12}\,|1\rangle\langle 2|.$$

We have $U\,|2\rangle = |2\rangle - \frac{i}{\hbar}tH_{12}\,|1\rangle$ which has norm $\|U\,|2\rangle\|^2 = 1 + \frac{t^2 H_{12}^2}{\hbar^2}$. The normalization is clearly increasing with time, so that the total probability is not conserved. This is because H is not Hermitian, such that the time evolution operator is not unitary.

3. Consider a free particle in one dimension, with $H = p^2/2m$. Evaluate $[x(t), x(0)]$ in the Heisenberg picture.

The Heisenberg equation of motion

$$-i\hbar\frac{\partial}{\partial t}x(t) = \frac{1}{2m}[p^2, x] = -\frac{1}{2m}[x, p^2] = -\frac{1}{2m}2i\hbar p$$

can be solved as

$$x(t) = x(0) + pt/m$$

where we note that p is time independent because it commutes with H. We then have

$$[x(t), x(0)] = [pt/m, x(0)] = -i\hbar t/m$$

4. Compute the Heisenberg equations of motion for x and p for a particle with Hamiltonian $H = \frac{p^2}{2m} + V(x)$. Compare to the classical equations of motion, $F = ma$ and $p = mv$.

Starting with $x(t)$:

$$-i\hbar\frac{\partial}{\partial t}x(t) = \frac{1}{2m}[p^2, x] = -\frac{1}{2m}[x, p^2] = -\frac{1}{2m}2i\hbar p$$

which implies

$$p = m\frac{\partial}{\partial t}x = mv$$

as expected from classical mechanics. Now computing $p(t)$:

$$-i\hbar\frac{\partial}{\partial t}p(t) = [V(x), p] = i\hbar V'(x).$$

Now recall that force is related to the gradient of potential: $F = -V'(x)$. We therefore have

$$F = -V'(x) = ma = \frac{\partial}{\partial t}p = m\frac{\partial^2}{\partial t^2}x = ma$$

as an operator equation.

5. Consider two eigenstates $|a_1\rangle$ and $|a_2\rangle$ of a Hermitian operator A. Assume that the eigenvalues are distinct, $a_1 \neq a_2$. Suppose the Hamiltonian is

$$H = |a_1\rangle\,\delta\,\langle a_2| + |a_2\rangle\,\delta\,\langle a_1|$$

where δ is a real number.

(a) Write down the energy eigenstates of the Hamiltonian and their energy eigenvalues

We note that $|a_{1,2}\rangle$ are orthonormal because they are eigenstates of a Hermitian operator with distinct eigenvalues. By inspection we see that

$$|a\pm\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle \pm |a_2\rangle)$$

are normalized eigenstates with eigenvalue $\pm\delta$ of H.

(b) Suppose the state is $|a_1\rangle$ at time $t = 0$. Find the state at other times within the Schrodinger picture.

We can write $|a_1\rangle = \frac{1}{\sqrt{2}}(|a_+\rangle + |a_-\rangle)$. Acting with the time evolution operator:

$$\begin{aligned} e^{-iHt/\hbar}|a_1\rangle &= \frac{1}{\sqrt{2}}(e^{-i\delta t/\hbar}|a_+\rangle + e^{+i\delta t/\hbar}|a_-\rangle) \\ &= \frac{1}{2}\left(e^{-i\delta t/\hbar}(|a_1\rangle + |a_2\rangle) + e^{i\delta t/\hbar}(|a_1\rangle - |a_2\rangle)\right) \\ &= \cos(\delta t/\hbar)|a_1\rangle - i\sin(\delta t/\hbar)|a_2\rangle \end{aligned}$$

(c) What is the probability of finding the state in $|a_2\rangle$ at time t if the state is $|a_1\rangle$ at time $t = 0$?

The probability is given by

$$\left| \langle a_2| e^{-iHt/\hbar} |a_1\rangle \right|^2 = |\sin(\delta t/\hbar)|^2$$

6. Consider a one dimensional simple harmonic oscillaor with annihilation operator

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right).$$

and the usual relationships $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle$.

(a) Evaluate $\langle m| x |n\rangle$, $\langle m| p |n\rangle$, $\langle m| \{x, p\} |n\rangle$, $\langle m| x^2 |n\rangle$, and $\langle m| p^2 |n\rangle$

Defining $l^2 = \hbar/m\omega$ we have

$$x = \frac{l}{\sqrt{2}}(a + a^\dagger), \qquad p = \frac{\hbar}{il\sqrt{2}}(a - a^\dagger)$$

so that

$$\langle m| x |n\rangle = \frac{l}{\sqrt{2}}(\langle m| a |n\rangle + \langle m| a^\dagger |n\rangle) = \frac{l}{\sqrt{2}}(\sqrt{n}\delta_{n,m-1} + \sqrt{m}\delta_{n,m+1}).$$

Similarly we have

$$\langle m| p |n\rangle = \frac{\hbar}{il\sqrt{2}}(\sqrt{n}\delta_{n,m+1} - \sqrt{m}\delta_{n,m-1})$$

We now use the symmetry of the anticommutator to show

$$\begin{aligned}\langle m| \{x, p\} |n\rangle &= \frac{\hbar}{2i} \langle m| \{a + a^\dagger, a - a^\dagger\} = \frac{\hbar}{i}(a^2 - (a^\dagger)^2 + \{a^\dagger, a\} - \{a, a^\dagger\}) |n\rangle \\ &= -i\hbar \langle m| a^2 - (a^\dagger)^2 |n\rangle = -i\hbar(\sqrt{n}\sqrt{m+1}\delta_{n,m+2} - \sqrt{m}\sqrt{n+1}\delta_{n,m-2}).\end{aligned}$$

Finally we have

$$\begin{aligned}\langle m| x^2 |n\rangle &= \frac{l^2}{2} \langle m| (a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2) |n\rangle \\ &= \frac{l^2}{2}(\sqrt{n}\sqrt{m+1}\delta_{n,m+2} + \sqrt{m}\sqrt{n+1}\delta_{n,m-2} + (2n+1)\delta_{m,n})\end{aligned}$$

and

$$\begin{aligned}\langle m| p^2 |n\rangle &= -\frac{\hbar^2}{2l^2} \langle m| (a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2) |n\rangle \\ &= \frac{\hbar^2}{2l^2}((2n+1)\delta_{m,n} - \sqrt{n}\sqrt{m+1}\delta_{n,m+2} + \sqrt{m}\sqrt{n+1}\delta_{n,m-2})\end{aligned}$$

(b) Check that the quantum version of the "virial theorem" holds, i.e.

$$\left\langle \frac{p^2}{m} \right\rangle = \left\langle x\frac{dV}{dx} \right\rangle$$

where the expectation value is taken in an energy eigenstate $|n\rangle$.

Using the results from the last part and $x\frac{dV}{dx} = m\omega^2 x^2$ we have

$$\left\langle \frac{p^2}{m} \right\rangle_n = \frac{\hbar^2}{2l^2 m}(2n+1) = \hbar\omega\left(n + \frac{1}{2}\right)$$

and

$$\left\langle x\frac{dV}{dx} \right\rangle_n = \left\langle m\omega^2 x^2 \right\rangle_n = \frac{l^2 m\omega^2}{2}(2n+1) = \hbar\omega\left(n + \frac{1}{2}\right)$$

such that the expectation values are equal.

7. Consider two sets of simple harmonic oscillators described by annihilation operators $a_\pm$ and creation operators $a_\pm^\dagger$. Let us define

$$J_\pm = \hbar a_\pm^\dagger a_\mp, \quad J_z = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \quad N = a_+^\dagger a_+ + a_-^\dagger a_-$$

as well as

$$\boldsymbol{J}^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+).$$

Prove that

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [\boldsymbol{J}^2, J_z] = 0, \quad J^2 = \frac{\hbar^2}{2}N(N/2+1)$$

We begin with the first relation, and use $[a^\dagger a, a^\dagger] = a^\dagger$ as well as $[a^\dagger a, a] = a$ for both sets of oscillators:

$$[J_z, J_+] = \frac{1}{2}\hbar^2[a_+^\dagger a_+ - a_-^\dagger a_-, \hbar a_+^\dagger a_-] = \frac{1}{2}\hbar^2(a_+^\dagger a_- + a_+^\dagger a_-) = \hbar J_+$$

and

$$[J_z, J_-] = \frac{1}{2}\hbar^2[a_+^\dagger a_+ - a_-^\dagger a_-, \hbar a_-^\dagger a_+] = \frac{1}{2}\hbar^2(-a_-^\dagger a_+ - a_-^\dagger a_+) = -\hbar J_-$$

so that $J_\pm = \hbar a_\pm^\dagger a_\mp$. We see that J_+ and J_- act as raising and lowering operators for J_z. This implies that J_z commutes with both $J_+ J_-$ and $J_- J_+$,

$$[J_z, J_\pm J_\mp] = [J_z, J_\pm]J_\mp + J_\pm[J_z, J_\mp] = \pm J_\pm J_\mp \mp J_\pm J_\mp = 0.$$

Therefore J_z commutes with each of the three terms in J^2 such that

$$[J_z, J^2] = 0.$$

We note that a complete set of oscillator states is given by $|n_+, n_-\rangle$ which we will abbreviate as $|n_\pm\rangle$. Note that we have

$$N\,|n_\pm\rangle = (n_+ + n_-)\,|n_\pm\rangle\,, \qquad J_z\,|n_\pm\rangle = \frac{\hbar}{2}(n_+ - n_-)\,|n_\pm\rangle$$

We also note that

$$J_\pm J_\mp\,|n_+, n_-\rangle = \hbar^2(a_\pm^\dagger a_\pm)(a_\mp a_\mp^\dagger)\,|n_+ n_-\rangle = \hbar^2 n_\pm(n_\mp + 1)\,|n_+, n_-\rangle$$

such that

$$J^2 |n_\pm\rangle = \hbar^2 \left(\frac{1}{4}(n_+ - n_-)^2 + \frac{1}{2}(n_+(n_- + 1) + n_-(n_+ + 1)) \right) |n_\pm\rangle$$
$$= \frac{\hbar^2}{4}(n_+^2 + 2n_+n_- + n_-^2 + 2n_+ + 2n_-) |n_\pm\rangle = \frac{\hbar^2}{4}(N^2 + 2N) |n_\pm\rangle = \frac{\hbar^2}{2} N(N/2 + 1) |n_\pm\rangle .$$

Since $|n_\pm\rangle$ form a complete basis, and we have showed that J^2 and $\frac{\hbar^2}{2}N(N/2+1)$ have the same action on this basis, we can identify them as operators.