Phys 251A Problem Set 4
Date posted: September 29, 2023
Due date: October 5, 2023

1. Consider the coherent state of the one-dimensional simple harmonic oscillator

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle$$

where α is a complex number and $a^\dagger$ is the Harmonic oscillator creation operator

(a) Show that $a|\alpha\rangle = \alpha|\alpha\rangle$ directly by acting on the above expression (hint: what is the value of the commutator $[a, f(a^\dagger)]$, for an arbitrary power-series-expandable f?)

> We will show that $[a, (a^\dagger)^n] = (n-1)(a^\dagger)^{n-1}$. Note that $[a, a^\dagger] = 1$ as a base case, and with the inductive hypothesis $[a, (a^\dagger)^{n-1}] = (n-2)(a^\dagger)^{n-2}$ we have
>
> $$[a, (a^\dagger)^n] = [a, (a^\dagger)^{n-1}]a^\dagger + [a, a^\dagger](a^\dagger)^{n-1} = (n-1)(a^\dagger)^{n-1}.$$
>
> Applying the above term by term to a power series expansion of $f(a^\dagger)$ thus yields
>
> $$[a, f(a^\dagger)] = f'(a^\dagger).$$
>
> Now turning to the coherent state and applying the above result to $f(a^\dagger) = e^{\alpha a^\dagger}$:
>
> $$a|\alpha\rangle = e^{-|\alpha|^2/2} a e^{\alpha a^\dagger} |0\rangle = e^{-|\alpha|^2/2}[a, e^{\alpha a^\dagger}]|0\rangle = \alpha e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle = \alpha|\alpha\rangle$$

(b) Using the completeness property of coherent states, $1 = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha$, define a "coherent state wavefunction" $\psi(\alpha)$ and derive the action of $a^\dagger$ on $\psi(\alpha)$.

> We insert the completeness relation
>
> $$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha|\psi\rangle = \frac{1}{\pi}\int d^2\alpha \psi(\alpha)|\alpha\rangle$$
>
> where we have defined the coherent state wavefunction $\psi(\alpha) = \langle\alpha|\psi\rangle$ analogously to the usual position space wavefunction. Through direct differentiation we find
>
> $$\frac{d}{d\alpha}|\alpha\rangle = \left(a^\dagger - \frac{\overline{\alpha}}{2}\right)|\alpha\rangle$$
>
> where the second term comes from the gaussian normalization factor We now act with $a^\dagger$:
>
> $$a^\dagger|\psi\rangle = \frac{1}{\pi}\int d^2\alpha \psi(\alpha) a^\dagger|\alpha\rangle = \frac{1}{\pi}\int d^2\alpha \psi(\alpha)\left(\frac{d}{d\alpha} + \frac{\overline{\alpha}}{2}\right)|\alpha\rangle = \frac{1}{\pi}\int d^2\alpha \left(-\frac{d}{d\alpha} + \frac{\overline{\alpha}}{2}\right)\psi(\alpha)|\alpha\rangle$$
>
> so that $a^\dagger$ acts on the wavefunction as
>
> $$\psi(\alpha) \to \left(-\frac{d}{d\alpha} + \frac{\overline{\alpha}}{2}\right)\psi(\alpha),$$
>
> analogously to how the momentum operator differentiates the position-space wavefunction with respect to x.
>
> One may notice another solution to the problem by inserting the completeness relation on the left of

the equation, then notice $\langle\alpha| a^\dagger = \langle\alpha| \alpha^*$:

$$\begin{aligned} I\, a^\dagger |\psi\rangle &= \int d^2\alpha \frac{1}{\pi} |\alpha\rangle \langle\alpha| a^\dagger |\psi\rangle \\ &= \int d^2\alpha \frac{1}{\pi} |\alpha\rangle \langle\alpha| \alpha^* |\psi\rangle \\ &= \int d^2\alpha \frac{1}{\pi} \alpha^* \psi(\alpha) |\alpha\rangle \end{aligned} \tag{1}$$

So the action of $a^\dagger$ on $\psi(\alpha)$:

$$a^\dagger : \psi(\alpha) \to \alpha^* \psi(\alpha) \tag{2}$$

we can prove that the two solutions we obtained earlier are equivalent. We just need to prove:

$$\int d^2\alpha \frac{1}{\pi} \big((\frac{\alpha^*}{2} - \frac{d}{d\alpha}) \psi(\alpha) \big) |\alpha\rangle = \int d^2\alpha \frac{1}{\pi} \alpha^* \psi(\alpha) |\alpha\rangle \tag{3}$$

which would be equivalent to show the following function equals zero:

$$\begin{aligned} &(\frac{\alpha^*}{2} + \frac{d}{d\alpha}) \psi(\alpha) \\ =&(\frac{\alpha^*}{2} + \frac{d}{d\alpha}) \langle\alpha|\psi\rangle \\ =&(\frac{\alpha^*}{2} + \frac{d}{d\alpha}) e^{-|\alpha|^2/2} \langle 0| e^{\alpha^* a} |\psi\rangle \\ =&\big((\frac{\alpha^*}{2} + \frac{d}{d\alpha}) e^{-\alpha\alpha^*/2}\big) \langle 0| e^{\alpha^* a} |\psi\rangle \\ =&0 \end{aligned} \tag{4}$$

2. Again focusing on coherent states, consider the operator $T_\alpha = e^{\alpha a^\dagger - \overline{\alpha} a}$.

   (a) Show that T_α generalizes both boosts and translations (i.e. for which α does T_α correspond to a boost, and for which α does T_α correspond to a translation?).

Using

$$a = \frac{1}{\sqrt{2}l}(x + i\frac{l^2}{\hbar}p), \qquad a^\dagger = \frac{1}{\sqrt{2}l}(x - i\frac{l^2}{\hbar}p)$$

we see that if $\alpha = -\overline{\alpha} = i\eta$, where η is then real, we obtain

$$T_\alpha = \exp(\alpha(a^\dagger + a)) = \exp(i\eta\sqrt{2}x/l)$$

which is a boost $p \to p + \hbar\sqrt{2}\eta/l$. Similarly if $\alpha = \overline{\alpha}$ we have

$$T_\alpha = \exp(\alpha(a^\dagger - a)) = \exp(-i\alpha\sqrt{2}lp/\hbar)$$

which is a translation by $\alpha\sqrt{2}l$.

   (b) Show that $T_\alpha |\beta\rangle$ is proportional to $|\beta + \alpha\rangle$. You may want to use Baker-Campbell-Hausdorff which yields a series expansion of Z in terms of X and Y where $e^Z = e^X e^Y$. Due to the oscillator algebra, the series truncates in this case: $Z = X + Y + \frac{1}{2}[X, Y] + \ldots$. You may also want to use the adjoint expansion $e^A B e^{-A} = e^{\mathrm{ad}_A} B$ and results from the problem (1)

By BCH we have

$$T_\alpha = e^{\alpha a^\dagger - \overline{\alpha} a} = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\overline{\alpha} a} = e^{|\alpha|^2/2} e^{-\overline{\alpha} a} e^{\alpha a^\dagger}$$

so that

$$|\beta\rangle = e^{-|\beta|^2/2} e^{\beta a^\dagger} |0\rangle = e^{-|\beta|^2/2} e^{\beta a^\dagger} e^{-\overline{\beta} a} |0\rangle = T_\beta |0\rangle .$$

We must therefore show that $T_\alpha T_\beta$ is proportional to $T_{\alpha+\beta}$; if we show this then it is clear that $T_\alpha |\beta\rangle = T_\alpha T_\beta |0\rangle$ is proportional to $T_{\alpha+\beta} |0\rangle = |\alpha + \beta\rangle$. To do this we first calculate the commutator of the terms inside the exponentials

$$[\alpha a^\dagger - \overline{\alpha} a, \beta a^\dagger - \overline{\beta} a] = -\alpha \overline{\beta} [a^\dagger, a] - \overline{\alpha} \beta [a, a^\dagger] = \alpha \overline{\beta} - \overline{\alpha} \beta$$

which is a number, so that BCH will terminate. We therefore have

$$T_\alpha T_\beta = T_{\alpha+\beta} e^{\frac{1}{2}(\alpha \overline{\beta} - \overline{\alpha} \beta)}$$

as desired

(c) From the previous part, conclude that $T_\alpha T_\beta = e^{i\phi} T_\beta T_\alpha$. Find ϕ in terms of α and β and interpret it in terms of an "area".

If $T_\alpha T_\beta$ is proportional to $T_{\alpha+\beta}$ we must have that $T_\beta T_\alpha$ is too. Indeed, a directly analogous BCH calculation leads to

$$T_\beta T_\alpha = T_{\alpha+\beta} e^{-\frac{1}{2}(\alpha \overline{\beta} - \overline{\alpha} \beta)}$$

so that

$$T_\alpha T_\beta = e^{i\phi} T_\beta T_\alpha, \qquad \phi = -i(\alpha \overline{\beta} - \overline{\alpha} \beta) = 2\,\mathrm{Im}(\alpha \overline{\beta}).$$

which is twice the signed area of the parallelogram spanned by α and β, regarded as vectors in the complex plane

3. How many quantum mechanical states are there per unit volume for a free particle per unit energy? This object is usually called the "density of states" $g(E)$ such that the number of states in a large volume V in an energy window ΔE is

$$N = \int_{\Delta E} g(E) dE, \tag{5}$$

where N is typically proportional to volume V. In this problem we will compute $g(E)$ for a free particle, $H\psi(x, y, z) = \frac{-\hbar^2 \nabla^2}{2m} \psi(x, y, z)$.

(a) Let us consider a finite box of size $L \times L \times L$ with periodic boundary conditions: $\psi(x + L, y + L, z + L) = \psi(x, y, z)$. Find the energy eigenstates of H and their energies.

The eigenstates are of the form $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ with eigenvalue $\frac{\hbar^2 |\boldsymbol{k}|^2}{2m}$ since these are the eigenstates of the momentum operator, and the Hamiltonian only depends on momentum. To satisy the boundary conditions we must have $\boldsymbol{k} = \frac{2\pi}{L} \boldsymbol{n}$, where $\boldsymbol{n}$ is a vector of integers.

(b) Consider a small window of energies dE, but even larger L so that $dE \gg 1/L$ and there are many states within the energy window dE. Compute the number of states dN that lie in the energy window dE as a function of L and E, and thus the density of states $g(E) = dN/dE$.

We will treat $\boldsymbol{n}$ as a continuous variable in the large L limit, as there are many states within the presumed energy window dE. We then have $dN = 4\pi|\boldsymbol{n}|^2 d|\boldsymbol{n}|$, where the prefactor comes from the surface area of the sphere of states with the same energy. Using that $dE = \hbar^2|\boldsymbol{k}|d|\boldsymbol{k}|/m$ and $d|\boldsymbol{k}| = \frac{2\pi}{L}d|\boldsymbol{n}|$ we have

$$\frac{dN}{dE} = 4\pi|\boldsymbol{n}|^2\frac{d|\boldsymbol{n}|}{dE} = 4\pi\frac{L^3}{(2\pi)^3}|\boldsymbol{k}|^2\frac{d|\boldsymbol{k}|}{dE} = L^3\frac{\sqrt{2}m^{3/2}\sqrt{E}}{\hbar^3 2\pi^2}$$

(c) What is $g(E)$ for a particle in one dimension instead? Two dimensions?

In two dimensions we instead have $dN = 2\pi|\boldsymbol{n}|d|\boldsymbol{n}|$ such that

$$\frac{dN}{dE} = 2\pi\frac{L^2}{(2\pi)^2}|\boldsymbol{k}|\frac{d|\boldsymbol{k}|}{dE} = L^2\frac{m}{2\pi\hbar^2},$$

such that the increasing velocity with energy precisely cancels the increased phase space associated with the circle of $\boldsymbol{k}$ with the same energy. In one dimension, we have $dN = 2d|\boldsymbol{n}|$, reflecting the two momenta with the same energy, such that

$$\frac{dN}{dE} = 2\frac{L}{2\pi}\frac{d|\boldsymbol{k}|}{dE} = \frac{L}{\pi\hbar}\sqrt{\frac{m}{2E}},$$

where we see that the DOS now decreases with energy because there is no geometric factor that overcomes the fact that energy levels are spaced further apart due to larger velocity ($= dE/d|\boldsymbol{k}|$) at higher energies.

4. Consider a Dirac particle in two dimensions. The Hamiltonian is

$$H = cp_x\sigma_x + cp_y\sigma_y + mc^2\sigma_z$$

where $\sigma_{x,y,z}$ are the Pauli matrices and satisfy $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and c is the speed of particles when $p \gg mc$.

(a) Compute the energy eigenvalues and the energy eigenstates of H

Note that the Hamiltonian commutes with the momenta p_x and p_y. We therefore include $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ in the eigenstate to diagonalize the momenta and partially diagonalize the Hamiltonian. The full eigenstate can be written as $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\begin{pmatrix} a \\ b \end{pmatrix}$ where we must solve for a and b. We then have

$$He^{i\boldsymbol{k}\cdot\boldsymbol{r}}\begin{pmatrix} a \\ b \end{pmatrix} = e^{ikx}(\hbar ck_x\sigma_x + \hbar ck_y\sigma_y + mc^2\sigma_z)\begin{pmatrix} a \\ b \end{pmatrix}.$$

This is now the same problem as a spin in a magnetic field: $\boldsymbol{b}\cdot\boldsymbol{\sigma}$ where $\boldsymbol{b} = (ck_x, ck_y, mc^2\sigma_z)$. We therefore have the eigenstates

$$\psi_{\boldsymbol{k}+} = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\begin{pmatrix} \cos(\theta_{\boldsymbol{k}}/2) \\ \sin(\theta_{\boldsymbol{k}}/2)e^{i\phi_{\boldsymbol{k}}} \end{pmatrix}, \qquad \psi_{\boldsymbol{k}-} = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\begin{pmatrix} \sin(\theta_{\boldsymbol{k}}/2) \\ -\cos(\theta_{\boldsymbol{k}}/2)e^{i\phi_{\boldsymbol{k}}} \end{pmatrix},$$

with corresponding eigenvalues

$$E_{\boldsymbol{k}\pm} = \pm|\boldsymbol{b}| = \pm\sqrt{\hbar^2c^2|\boldsymbol{k}|^2 + m^2c^4}$$

above we have used the parameterization

$$\cos(\theta) = b_z/|\boldsymbol{b}| = mc^2/E_{\boldsymbol{k}}, \qquad e^{i\phi_{\boldsymbol{k}}} = \frac{k_x + ik_y}{|\boldsymbol{k}|}.$$

We have also used the transformations $\theta_{\boldsymbol{k}} \to \pi - \theta_{\boldsymbol{k}}$ and $\phi_{\boldsymbol{k}} \to \phi_{\boldsymbol{k}} + \pi$ in order to invert the direction of the spinor relative to $\boldsymbol{b}$ which obtains $\psi_{\boldsymbol{k}-}$ from $\psi_{\boldsymbol{k}+}$.

(b) Calculate the density of states $g(E) = dN/dE$ for $m = 0$

(c) Calculate the density of states for generic m and compare your answer to an ordinary $p^2/2m$ particle in two-dimensions for $cp \ll mc^2$.

Like in previous problem, we impose periodic boundary conditions and quantize $\boldsymbol{k}$ appropriately. The only difference is now $dE/d|\boldsymbol{k}| = c^2\hbar^2|\boldsymbol{k}|/E_{\boldsymbol{k}}$, for any m as long as $E_{\boldsymbol{k}} \geq |m|c^2$ is a valid energy, so that the DOS is

$$\frac{dN}{dE} = 2\pi \frac{L^2}{(2\pi)^2}|\boldsymbol{k}|\frac{d|\boldsymbol{k}|}{dE} = \frac{L^2}{2\pi\hbar^2 c^2} E\Theta(|E| - m)$$

where we have added a Heaviside theta function to reflect that the DOS is zero if $|E_{\boldsymbol{k}}| < mc^2$. We see that if $m = 0$, the DOS increases linearly with energy. If m is nonzero, then the DOS is zero until $|E| > mc^2$ after which DOS grows linearly with the *total* E.

When $cp \ll mc^2$ and $E > mc^2$ so that the DOS is nonzero, we have $E \approx mc^2$ so that

$$\frac{dN}{dE} \approx \frac{L^2}{2\pi\hbar^2} m$$

which matches the nonrelativistic result in two dimensions.