Phys 251A Problem Set 5
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1. Consider two Hamiltonians $H_1 = A^\dagger A$ and $H_2 = AA^\dagger$ for some operator A that is not necessarily Hermitian.

   (a) Show that H_1 and H_2 have a non-negative spectrum (all eigenvalues are non-negative).

   We calculate, for a normalized eigenvector ψ_1 of H_1 with eigenvalue λ,

   $$\lambda = \langle\psi_1| H |\psi_1\rangle = \langle\psi_1| A^\dagger A |\psi_1\rangle = \| A |\psi_1\rangle \|^2$$

   which is clearly non-negative. The proof is identical for H_2 with $A \leftrightarrow A^\dagger$.

   (b) Show that H_1 and H_2 have the same spectrum at nonzero energies by relating their eigenvectors.

   Suppose $H_1 |\psi_1\rangle = A^\dagger A |\psi_1\rangle = \lambda_1 |\psi_1\rangle$, where $\lambda_1 \neq 0$. Then, acting to the left by A we have

   $$AA^\dagger A |\psi_1\rangle = A\lambda_1 |\psi_1\rangle \quad \Longrightarrow \quad H_2 A |\psi_1\rangle = \lambda_1 A |\psi_1\rangle$$

   such that $A|\psi_1\rangle$ is an eigenvector of H_2 of the same eigenvalue. Likewise, the eigenstates of H_2 can be mapped onto those of H_1 by the action of $A^\dagger$ such that the two spectra are in one-to-one correspondence.

2. In this problem we will use the tools developed in problem (1) solve the spectrum of following Hamiltonian exactly.

   $$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{2V_0}{\cos^2(x/a)}, \qquad V_0 = \frac{\hbar^2}{2ma^2}$$

   Here the particle is confined to lie in the interval $-\pi a/2 < x < \pi a/2$. At the endpoints we impose $\psi(x = \pm\pi a/2) = 0$ in accordance with the fact that the potential is infinite at these points.

   (a) Rewrite the above problem in a non-dimensionalized form, with $y = x/a$ and $H_0 = H/V_0$.

   Using $\frac{d^2}{dx^2} = \frac{1}{a^2}\frac{d^2}{dy^2}$, we have

   $$H_0 = \frac{H}{V_0} = -\frac{d^2}{dy^2} + \frac{2}{\cos^2 y}$$

   (b) Define the operator $A = \frac{d}{dy} + \tan y$ and compute the Hamiltonians $H_1 = A^\dagger A$ and $H_2 = AA^\dagger$. Recall that $\left(\frac{d}{dy}\right)^\dagger = -\frac{d}{dy}$.

   We have

   $$\begin{aligned} H_1 = A^\dagger A &= -\frac{d^2}{dy^2} - \frac{d}{dy}\tan(y) + \tan(y)\frac{d}{dy} + \tan^2 y \\ &= -\frac{d^2}{dy^2} - [\frac{d}{dy}, \tan y] + \tan^2 y = -\frac{d^2}{dy^2} - \sec^2 y + \tan^2 y \\ &= -\frac{d^2}{dy^2} - 1 \end{aligned}$$

and

$$\begin{aligned}H_2 = AA^\dagger &= -\frac{d^2}{dy^2} + \frac{d}{dy}\tan(y) - \tan(y)\frac{d}{dy} + \tan^2 y \\ &= -\frac{d^2}{dy^2} + [\frac{d}{dy}, \tan y] + \tan^2 y = -\frac{d^2}{dy^2} + \sec^2 y + \tan^2 y \\ &= -\frac{d^2}{dy^2} + \frac{2}{\cos^2 y} - 1\end{aligned}$$

(c) One of H_1 and H_2 is particularly simple. Solve for its entire spectrum.

H_1 is a free particle Hamiltonian up to a constant. The eigenstates are labeled by positive integers $n = 1, 2, \ldots$. The eigenstates of even parity are $\cos ny$, where n is odd, and the eigenstates of odd parity are $\sin ny$, where n is even. Either way, the action of $-\frac{d^2}{dy^2}$ on these states is n^2, so that the eigenvalues are $n^2 - 1$

(d) Show that the more complicated Hamiltonian has no zero energy states that satisfy the boundary condition (hint: use that $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$)

If $H_2\psi = AA^\dagger\psi = 0$ then we have $\langle\psi| H |\psi\rangle = \left\|A^\dagger\psi\right\|^2$ such that $A^\dagger\psi = 0$ or

$$\frac{d}{dy}\psi = \tan(y)\psi.$$

The solution $\psi(y) = C\sec(y)$ is unique up to a constant C, because this is an ordinary differential equation. But $\sec(y)$ blows up at the boundaries, let alone vanishing. So there is no allowable zero energy state.

(e) Use problem (1) of this problem set to deduce the entire spectrum of both $H_{1,2}$.

We have already determined the spectrum of H_1. By problem 1 we can act with A on an eigenstate of H_1 to obtain an eigenstate of H_2. This works for all states except for $\cos(y)$, which is a zero mode of A. So the eigenvalues of H_2 are $n^2 - 1$ for $n \geq 2$. The eigenstates are H_2 are then

$$\begin{cases} A\sin(ny) = n\cos(ny) + \tan(y)\sin(ny) & \text{for } n = 2, 4, 6, \ldots \\ A\cos(ny) = -n\sin(ny) + \tan(y)\cos(ny) & \text{for } n = 3, 5, 7, \ldots \end{cases}$$

(f) Find the eigenstates and eigenvalues of H_0 and its dimensionalized version H.

$H_0 = H_2 + 1$ so the eigenstates are the same as those of H_2 and the eigenvalues are n^2 for $n \geq 2$. The eigenstates of H are also the same; written in terms of x they are

$$\begin{cases} n\cos(nx/a) + \tan(x/a)\sin(nx/a) & \text{for } n = 2, 4, 6, \ldots \\ -n\sin(nx/a) + \tan(x/a)\cos(nx/a) & \text{for } n = 3, 5, 7, \ldots \end{cases}$$

and the eigenvalues are $V_0 n^2$ for $n \geq 2$.

3. Consider a charged particle, in two dimensions, in a position independent magnetic field $B = \boldsymbol{\nabla} \times \boldsymbol{A} = \partial_x A_y - \partial_y A_x$. Note that $\boldsymbol{A}$ must be position dependent. The Hamiltonian is given by

$$H = \frac{\boldsymbol{\pi}^2}{2m}, \qquad \boldsymbol{\pi} = -i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}.$$

where $\boldsymbol{\pi}$ is the "dynamical momentum" in the presence of the magnetic field.

(a) Compute the Heisenberg equations of motion for x and y and physically justify the labeling of $\boldsymbol{\pi}$ as the "dynamical momentum."

$$i\hbar\frac{d}{dt}x = \frac{1}{2m}[x, \boldsymbol{\pi}^2] = \frac{1}{2m}[x, \boldsymbol{\pi}] \cdot \boldsymbol{\pi} + \boldsymbol{\pi} \cdot [x, \boldsymbol{\pi}] = \frac{1}{m} i\hbar\pi_x$$

and similarly for y such that

$$\frac{d}{dt}\boldsymbol{r} = \frac{1}{m}\boldsymbol{\pi}$$

where $\boldsymbol{r} = (x, y)$. So $\boldsymbol{\pi}$ is mass times velocity, which justifies labeling it "dynamical momentum"

(b) Define a gauge transformation of an operator "M" as $M \to e^{-ie\phi(\boldsymbol{r})/\hbar} M e^{ie\phi(\boldsymbol{r})/\hbar}$, where M acts on wavefunctions $\psi(\boldsymbol{r})$. Show that, under a gauge transformation, the dynamical momentum $\boldsymbol{\pi}$ is replaced by a similar dynamical momentum except with $\boldsymbol{A}$ replaced by $\boldsymbol{A} - \boldsymbol{\nabla}\phi(\boldsymbol{r})$ (which does not change the magnetic field).

$$\begin{aligned}\boldsymbol{\pi} \to e^{-ie\phi(\boldsymbol{r})/\hbar}(-i\hbar\boldsymbol{\nabla} - e\boldsymbol{A})e^{ie\phi(\boldsymbol{r})/\hbar} &= e^{-ie\phi(\boldsymbol{r})/\hbar}(e^{ie\phi(\boldsymbol{r})/\hbar}(-i\hbar\boldsymbol{\nabla}) + [-i\hbar\boldsymbol{\nabla}, e^{ie\phi(\boldsymbol{r})/\hbar}] - e\boldsymbol{A}) \\ &= -i\hbar\boldsymbol{\nabla} - e(\boldsymbol{A} - \boldsymbol{\nabla}\phi(\boldsymbol{r}))\end{aligned}$$

as desired

(c) Compute $[\pi_x, \pi_y]$ in terms of the magnetic field B.

$$[\pi_x, \pi_y] = [-i\hbar\partial_x - eA_x, -i\hbar\partial_y - eA_y] = i\hbar e(\partial_x A_y - \partial_y A_x) = i\hbar eB$$

(d) Compute the Heisenberg equations of motion for π_x and π_y and discuss them in the context of the classical Lorentz force law.

we have

$$i\hbar\frac{d}{dt}\pi_x = \frac{1}{2m}[\pi_x, \pi_y^2] = i\hbar eB\frac{\pi_y}{m}$$

and

$$i\hbar\frac{d}{dt}\pi_y = \frac{1}{2m}[\pi_y, \pi_x^2] = -i\hbar eB\frac{\pi_x}{m}$$

such that

$$\frac{d}{dt}\boldsymbol{\pi} = \frac{e}{m}\boldsymbol{\pi} \times (B\hat{\boldsymbol{z}})$$

which is the Lorentz force; force is equal to velocity cross producted with the magnetic field.

4. Charged particle in a field II

(a) Compare the commutation relation $[\pi_x, \pi_y]$ to that of x and p, and the charged particle Hamiltonian to that of the Harmonic oscillator Hamiltonian. Write down ladder operators such that

$$H = \frac{\boldsymbol{\pi}^2}{2m} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).$$

Compare to the so-called "cyclotron frequency" of classical mechanics. Eigenstates of $a^\dagger a$ here are called "Landau levels."

The commutation relations are very similar, except scaled up by eB. Without loss of generality we suppose that $eB > 0$. We can therefore construct ladder operatrs

$$a = \frac{1}{\sqrt{2\hbar eB}}(\pi_x + i\pi_y), \qquad a^\dagger = \frac{1}{\sqrt{2\hbar eB}}(\pi_x - i\pi_y)$$

such that $[a, a^\dagger] = 1$. For $eB < 0$, replace the prefactor with $1/\sqrt{2\hbar|eB|}$ and exchange the roles of a and $a^\dagger$. The Hamiltonian is given by

$$H = \hbar\omega(a^\dagger a + \frac{1}{2})$$

where $\omega = |eB|/m$ can be identified as the cyclotron frequency. In quantum mechanics, therefore, cyclotron orbits are quantized with a harmonic oscillator spectrum. The different quantized orbits, labeled by $n = 0, 1, 2, \ldots$, the eigenvalue of $a^\dagger a$, are known as "Landau levels."

(b) It seems that we have gone from a continuum of states to a discrete set as soon as we have included a magnetic field. This is not the case, however, and each Landau level has many states inside it. To see this, we will construct operators that act "within" a Landau level. Consider the operators

$$R_x = x - \lambda\pi_y, \qquad R_y = y + \lambda\pi_x$$

where x and y are the usual position operators in two dimensions. Solve for λ such that $[R_x, \boldsymbol{\pi}] = 0$ and $[R_y, \boldsymbol{\pi}] = 0$. Conclude that $[R_{x,y}, a^\dagger a] = 0$, such that R_x and R_y act "within" each Landau level (they can be simultaneously diagonalized with the number operator).

We have $[R_x, \pi_y] = [R_y, \pi_x] = 0$ regardless of λ, since, for example, both x and π_y commute with π_y. We therefore compute

$$[R_x, \pi_x] = [x, \pi_x] - \lambda[\pi_y, \pi_x] = i\hbar + i\hbar eB\lambda$$

and

$$[R_y, \pi_y] = [y, \pi_y] + \lambda[\pi_x, \pi_y] = i\hbar + i\hbar eB\lambda$$

such that we must choose $\lambda = -1/(eB)$. Because R_x and R_y commute with both π_x and π_y, they commute with the number operator $a^\dagger a$. Since they therefore do not change the eigenvalue of $a^\dagger a$, they act within a Landau level.

(c) Compute $[R_x, R_y]$. Write down ladder operators b and $b^\dagger$ that each commute with both of a and $a^\dagger$. Show that there are infinitely many states within a Landau level.

We have

$$[R_x, R_y] = [x + \pi_y/eB, y - \pi_x/eB] = -i\frac{\hbar}{eB}.$$

Then we can define, for example

$$b = \frac{\sqrt{eB}}{\sqrt{2\hbar}}(R_x - iR_y), \qquad b^\dagger = \frac{\sqrt{eB}}{\sqrt{2\hbar}}(R_x + iR_y),$$

for $eB > 0$ without loss of generality, such that $[b, b^\dagger] = 1$. There are many other choices of ladder operators as well; there is no Hamiltonian inside the Landau level here so there is nothing fixing the relative scaling between the “canonical pair” R_x and R_y other than symmetry.