

Phys 251A Problem Set 6

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1. In this problem we will compute some prefactors we neglected in lecture and, in the process, understand objects known as “functional determinants.” It is recommended to do the following two problems in order.

(a) Show that

$$\int_{-\infty}^{\infty} dx_1 \cdots dx_n e^{-\frac{1}{2} \mathbf{x}^T A \mathbf{x}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

where $\mathbf{x} = (x_1, \dots, x_n)^T$ and A is an $n \times n$ real, symmetric, positive-definite matrix. You should write your answer in terms of the determinant of A . To do the integral, you should perform a change of variables via an orthogonal matrix that diagonalizes A . To account for the change in the integration measure, you should use that the determinant of orthogonal matrix is ± 1 . Finally, you will want to use the fact that the determinant is the product of eigenvalues.

We make the change of variables $\mathbf{y} = R\mathbf{x}$ where $R^T = R$ is a symmetric matrix that diagonalizes A ,

$$RAR^T = \text{diag}(\lambda_1, \dots, \lambda_n),$$

where λ_i is the i 'th eigenvalue of A . The Jacobian of this change of variables is $|\det R| = 1$, where we used that the determinant of an orthogonal matrix is ± 1 . That is, the integration measure and limits have the same form when written in terms of y and x . We therefore obtain

$$\int_{-\infty}^{\infty} dy_1 e^{-\frac{1}{2} \lambda_1 y_1^2} \cdots dy_n e^{-\frac{1}{2} \lambda_n y_n^2} = \frac{(2\pi)^{n/2}}{\sqrt{\lambda_1 \cdots \lambda_n}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

(b) Calculate

$$\int_{-\infty}^{\infty} d^2 z_1 \cdots d^2 z_n e^{-\frac{1}{2} \mathbf{z}^\dagger K \mathbf{z}}$$

in terms of $\det K$ where $\mathbf{z} = (z_1, \dots, z_n)^T$ is a vector of complex numbers and $K = K^\dagger$ is a positive definite Hermitian matrix. You should use a similar argument as in (a), suitably generalized to the complex case.

We take $\mathbf{w} = U\mathbf{z}$ where U is a unitary matrix such that $UKU^\dagger = \text{diag}(\lambda_1, \dots, \lambda_n)$, where the eigenvalues are real because K is Hermitian. Note that $d^2 z_i$ is proportional to $dz_i d\bar{z}_i$. We then see that $dw_1 \cdots dw_n = \det U dz_1 \cdots dz_n$ and $d\bar{w}_1 \cdots d\bar{w}_n = \det \bar{U} dz_1 \cdots dz_n$. We now use that the eigenvalues of unitary matrices are phases: $\lambda = e^{i\theta}$ where θ is real. Then the determinant is also a phase, so that $\det U \det \bar{U} = |\det U|^2 = 1$ and the integration measure again does not change. We then obtain

$$\int_{-\infty}^{\infty} d^2 z_1 e^{-\frac{1}{2} \lambda_1 |z_1|^2} \cdots d^2 z_n e^{-\frac{1}{2} \lambda_n |z_n|^2} = \frac{\pi^n}{\lambda_1 \cdots \lambda_n} = \frac{(2\pi)^n}{\det K}$$

(c) Recall the propagator that we calculated from the path integral in lecture

$$\langle x_f, T | x_i, 0 \rangle = \langle x_f | e^{-iHT} | x_i \rangle = C(T) \exp\left(\frac{im}{2\hbar} \frac{(x_f - x_i)^2}{T}\right) \quad (1)$$

where

$$C(T) = \int_{y(0)=0}^{y(T)=0} \mathcal{D}y \exp\left(\frac{im}{2\hbar} \int_0^T \dot{y}^2 dt\right) = \int_{y(0)=0}^{y(T)=0} \mathcal{D}y \exp\left(-\frac{m}{2\hbar} \int_0^{\tau_0} (\partial_\tau y)^2 d\tau\right).$$

In the second step we have changed variables to $\tau = it$ and defined $\tau_0 = iT$.

The normalization of $C(T)$ is subtle, because the path integral measure $\mathcal{D}y$ is tricky to define in a finite way. It is easier to deduce the correct normalization through comparing with the operator formalism.

Compute $C(T)$ by starting with

$$1 = \int dx' \langle x'|x \rangle$$

and inserting UU^\dagger inside the bracket, where $U = e^{-iHT}$. Insert a resolution of the identity between U and U^\dagger , and identify the two factors as propagators. Use the identity derived in lecture, (1), and perform the gaussian integrals to arrive at an expression for $C(T)$. Note that $C(T) > 0$ when $\tau_0 = iT$ is real and positive from the path integral definition; this can be used to fix the phase of $C(T)$.

If you wish, the integral can be calculated by assuming that τ_0 is real, and subsequently “analytically continuing” to imaginary τ_0 and real T . This simply amounts to substituting iT for τ_0 in the final expression and treating T as ordinary real time. Alternatively you can do the real time integral directly and argue that the rapid oscillations at infinity cancel out with each other, or use contour integral techniques to rotate the contour to be some, arbitrary, diagonal of the complex plane.

We have, promoting T to be a complex number,

$$\begin{aligned} 1 &= \int dx' \langle x'|x \rangle = \int dx' dx'' \langle x'| e^{iH\bar{T}} |x'' \rangle \langle x''| e^{-iHT} |x \rangle \\ &= |C(T)|^2 \int dx' dx'' \exp\left(\frac{-im}{2\hbar} \frac{(x' - x'')^2}{\bar{T}}\right) \exp\left(\frac{im}{2\hbar} \frac{(x'' - x')^2}{T}\right) \end{aligned}$$

Doing both Gaussian integrals, where T has a negative imaginary part for manifest convergence, then leads to

$$|C(T)|^{-2} = 2\pi\hbar \frac{|T|}{m}$$

We pause to comment that the small imaginary part of T is a useful trick that turns a convergent oscillatory integral into an absolutely convergent (integral of absolute value finite) integral without affecting the final result. This absolutely convergent integral can then be squared, and Fubini’s theorem used, to evaluate the gaussian integral in the standard way. Alternatively, one can use contour integral tricks or Wick rotation and analytic continuation.

Fixing the phase of $C(T)$ as in the problem description gives

$$C(T) = \sqrt{\frac{m}{2\pi i\hbar T}}$$

so that $C(T)$ is real when T is purely imaginary with negative imaginary part.

2. In this problem we will use one example to illustrate the adiabatic theorem. Consider the Hamiltonian

$$H(\mathbf{B}) = \frac{\hbar\gamma}{2} \mathbf{B} \cdot \boldsymbol{\sigma}$$

where the magnetic field $\mathbf{B}(t)$ is the changing very slowly as $\mathbf{B}(t) = B_0(\cos \phi(t), \sin \phi(t), 0)$, where $\phi(t) = 2\pi t/T$.

- (a) Find the instantaneous eigenstates $|\psi_{\pm}(t)\rangle$ and their eigenvalues $E_{\pm}(t)$ at time t . Show that the eigenvalues are time-independent.

The Hamiltonian takes the form

$$H = \frac{\hbar\gamma B_0}{2} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

The positive and negative eigenvalues are $E_+ = \frac{\hbar\gamma B_0}{2}$ and $E_- = -\frac{\hbar\gamma B_0}{2}$ with corresponding eigenstates:

$$|\psi_+(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}, \quad |\psi_-(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\phi} \end{pmatrix}$$

It is easy to see the eigenvalues are time-independent.

- (b) For an arbitrary state $|\alpha, t\rangle = c_+(t)|\psi_+(t)\rangle + c_-(t)|\psi_-(t)\rangle$, calculate the time derivative of the coefficients $\dot{c}_\pm(t)$ in terms of $c_\pm(t)$. Express your result as a 2×2 matrix transformation:

$$\begin{pmatrix} \dot{c}_+ \\ \dot{c}_- \end{pmatrix} = \mathbf{M}_{2 \times 2} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

Following the derivation in the lecture, we have

$$\dot{c}_m(t) = - \left[\frac{i}{\hbar} E_m(t) + \langle \psi_m(t) | \frac{d}{dt} | \psi_m(t) \rangle \right] c_m - \sum_{n \neq m} \frac{\langle \psi_m(t) | \dot{H}(t) | \psi_n(t) \rangle}{E_n(t) - E_m(t)} c_n$$

For our two energy levels system, the formula becomes

$$\begin{aligned} \dot{c}_+(t) &= - \left[\frac{i}{\hbar} \frac{\hbar\gamma B_0}{2} + \frac{1}{2} (1 \ e^{-i\phi}) \begin{pmatrix} 0 & \\ i\dot{\phi}e^{i\phi} & \end{pmatrix} \right] c_+ - \frac{1}{2} (1 \ e^{-i\phi}) \begin{pmatrix} 0 & -i\dot{\phi}e^{-i\phi} \\ i\dot{\phi}e^{i\phi} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -e^{i\phi} \end{pmatrix} \frac{c_-}{-\frac{\hbar\gamma B_0}{2} - \frac{\hbar\gamma B_0}{2}} \\ &= -\frac{i}{2}(\gamma B_0 + \frac{2\pi}{T})c_+ + \frac{i\pi}{T}c_- \end{aligned}$$

and

$$\dot{c}_-(t) = \frac{i}{2}(\gamma B_0 - \frac{2\pi}{T})c_- + \frac{i\pi}{T}c_+$$

So in the matrix form,

$$\begin{pmatrix} \dot{c}_+ \\ \dot{c}_- \end{pmatrix} = \begin{pmatrix} -\frac{i}{2}(\gamma B_0 + \frac{2\pi}{T}) & \frac{i\pi}{T} \\ \frac{i\pi}{T} & \frac{i}{2}(\gamma B_0 - \frac{2\pi}{T}) \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

- (c) Write $\Delta = |E_+ - E_-|$, show that when $T \gg \hbar/\Delta$, that is when the Hamiltonian is changing slowly compared with the intrinsic timescale of the system, we can ignore the off-diagonal terms. This is the adiabatic approximation.

The energy gap is $\Delta = \hbar\gamma B_0$. When the Hamiltonian changes slowly, $T \gg \hbar/\Delta = 1/\gamma B_0$, that is, $\gamma B_0 \gg 1/T$, it is easy to see the off-diagonal terms are much smaller compared with the diagonal terms. We can ignore the off-diagonal terms, that is to ignore the transition between different energy levels.

- (d) Calculate the Berry phase of the ground state accumulated from $t = 0$ to T , after which the Hamiltonian goes back to the initial values. Comment on your result: does the ground state also return to the initial value?

For the Berry phase of the ground state (with eigenvalue E_-), we have the formula:

$$\gamma_- = i \int_0^T dt \langle \psi_-(t) | \dot{\psi}_-(t) \rangle = i \int_0^T dt \frac{1}{2} (1 \ -e^{-i\phi}) \begin{pmatrix} 0 & \\ -i\dot{\phi}e^{i\phi} & \end{pmatrix} = i \int_0^T dt \frac{1}{2} i \frac{2\pi}{T} = -\pi$$

We find that the ground state does not go back to the initial value but accumulates a phase of $e^{-i\pi}$.

3. In the problem we will work out of the example of the Berry phase that was briefly discussed in class. Again consider the Hamiltonian

$$H(\mathbf{B}) = \frac{\hbar\gamma}{2} \mathbf{B} \cdot \boldsymbol{\sigma}$$

where the magnetic field \mathbf{B} is the changing external parameter. Writing $\mathbf{B} = B(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, where $B = |\mathbf{B}|$

- (a) Write down the Hamiltonian as a 2×2 matrix in terms of θ and ϕ , and find its ground state.

Plug in the matrix form

$$H = \frac{\hbar\gamma B}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

We have solved the eigensystem of this matrix in previous problem sets, the eigenvalues are $E_+(\mathbf{B}) = \frac{\hbar\gamma B}{2}$ and $E_-(\mathbf{B}) = -\frac{\hbar\gamma B}{2}$, their corresponding eigenstates are:

$$|+, \mathbf{B}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad |-, \mathbf{B}\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

as E_- has lower energy, so $|-, \mathbf{B}\rangle$ is the ground state. Notice that the eigenstates do not depend on the magnitude of field B .

- (b) Using the definition of Berry connection and Berry curvature,

$$\mathbf{A}_a(\mathbf{R}) = i \langle \psi_n(\mathbf{R}) | \frac{\partial}{\partial R^a} | \psi_n(\mathbf{R}) \rangle$$

$$\mathbf{F}_{ab} = \partial_a \mathbf{A}_b - \partial_b \mathbf{A}_a$$

compute the Berry connection \mathbf{A}_θ and \mathbf{A}_ϕ , and Berry curvature $\mathbf{F}_{\theta\phi}$

Berry connection of the ground state:

$$\mathbf{A}_\theta = i \langle -, \mathbf{B} | \partial_\theta | -, \mathbf{B} \rangle = i \left(\sin\frac{\theta}{2} \quad -\cos\frac{\theta}{2} e^{-i\phi} \right) \begin{pmatrix} \frac{1}{2} \cos\frac{\theta}{2} \\ \frac{1}{2} \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix} = 0$$

$$\mathbf{A}_\phi = i \langle -, \mathbf{B} | \partial_\phi | -, \mathbf{B} \rangle = i \left(\sin\frac{\theta}{2} \quad -\cos\frac{\theta}{2} e^{-i\phi} \right) \begin{pmatrix} 0 \\ -i \cos\frac{\theta}{2} e^{i\phi} \end{pmatrix} = -\cos^2\frac{\theta}{2} = -\frac{1 + \cos\theta}{2}$$

Berry curvature of the ground state:

$$\mathbf{F}_{\theta\phi} = \partial_\theta \mathbf{A}_\phi - \partial_\phi \mathbf{A}_\theta = \partial_\theta \left(-\frac{1 + \cos\theta}{2} \right) = \frac{1}{2} \sin\theta$$

- (c) To transform the Berry curvature to the Cartesian coordinates, use the coordinate transformation rule $\tilde{\mathbf{F}}_{\mu\nu} = T_\mu^\lambda T_\nu^\rho \mathbf{F}_{\lambda\rho}$, where $T_\nu^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}$, and

$$(|B|, \theta, \phi) = \left(\sqrt{B_x^2 + B_y^2 + B_z^2}, \arccos(B_z / \sqrt{B_x^2 + B_y^2 + B_z^2}), \arctan(B_y / B_x) \right)$$

Show the Berry curvature has the following form in Cartesian coordinates

$$\mathbf{F}_{ij} = \epsilon_{ijk} \frac{B_k}{2|B|^3}$$

You can use Mathematica or other tools for simplification.

Notice that because the eigenstates do not depend on $B = |\mathbf{B}|$, so $\mathbf{A}_B = 0$, and $\mathbf{F}_{B\theta} = \mathbf{F}_{B\phi} = 0$. In the Cartesian coordinates, for convenience, we write, $\mathbf{A}_x = i \langle -, \mathbf{B} | \partial_{B_x} | -, \mathbf{B} \rangle$ and similarly for other components. Additionally, $\mathbf{F}_{xy} = \partial_{B_x} \mathbf{A}_y - \partial_{B_y} \mathbf{A}_x$ and so on.

Using the coordinate transformation rule:

$$\mathbf{F}_{xy} = \frac{\partial \theta}{\partial B_x} \frac{\partial \phi}{\partial B_y} \mathbf{F}_{\theta\phi} + \frac{\partial \phi}{\partial B_x} \frac{\partial \theta}{\partial B_y} \mathbf{F}_{\phi\theta}$$

with $\mathbf{F}_{\theta\phi} = -\mathbf{F}_{\phi\theta} = \frac{1}{2} \sin \theta = \sqrt{1 - \frac{B_z^2}{B_x^2 + B_y^2 + B_z^2}}$, using Mathematica to simplify, we have,

$$\frac{\partial \theta}{\partial B_x} = \frac{B_x B_z}{(B_x^2 + B_y^2 + B_z^2)^{3/2} \sqrt{1 - \frac{B_z^2}{B_x^2 + B_y^2 + B_z^2}}}$$

and so on. Finally, the Berry curvature in Cartesian coordinates,

$$\mathbf{F}_{xy} = \frac{B_z}{2(B_x^2 + B_y^2 + B_z^2)^{3/2}}$$

and so on. Write in the Einstein notation,

$$\mathbf{F}_{ij} = \epsilon_{ijk} \frac{B_k}{2|B|^3}$$

- (d) Integrate the Berry curvature over the sphere where $|B|$ is fixed. Show that the result is an integer multiple of 2π . You may find that it is easier to do the integration in $\theta - \phi$ coordinates.

We first show the integration in Cartesian coordinates, fixing the outward orientation.

$$\int_{|B|_{const}} d\mathbf{S}^{ij} \mathbf{F}_{ij} = \int dS \frac{\hat{\mathbf{n}} \cdot \mathbf{B}}{2|B|^3} = 4\pi |B|^2 \frac{|B|}{2|B|^3} = 2\pi$$

Alternatively, we can integrate in the $\theta - \phi$ coordinates using result from part (b),

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{1}{2} \sin \theta = 2\pi$$

Either way, we have an integer multiple of 2π with Chern number $C = 1$

4. Consider a Dirac particle in a low energy band of a crystal with Hamiltonian

$$H = \hbar v \hat{k}_x \sigma_x + \hbar v \hat{k}_y \sigma_y + \Delta \sigma_z.$$

where the “band gap” Δ functions like a “mass.” Here, $\hat{\mathbf{k}} = -i\nabla$ is the wavevector, or “crystal momentum” operator. In condensed matter physics we often insert a complete set of momentum eigenstates so that

$$H = \sum_{\mathbf{k}} |\mathbf{k}\rangle H(\mathbf{k}) \langle \mathbf{k}|, \quad H(\mathbf{k}) = \langle \mathbf{k}| H |\mathbf{k}\rangle = \hbar v k_x \sigma_x + \hbar v k_y \sigma_y + \Delta \sigma_z.$$

We note that $H(\mathbf{k})$ is now a 2×2 Hamiltonian that depends on the *parameters*, not operators, k_x and k_y . Let us denote the positive energy eigenstate of $H(\mathbf{k})$ as $|\mathbf{k}+\rangle$ and the negative energy eigenstate as $|\mathbf{k}-\rangle$.

- (a) Compute the “Berry connection” for the positive eigenstates: $\mathbf{A} = i \langle \mathbf{k}+ | \nabla_{\mathbf{k}} | \mathbf{k}+ \rangle$, where the gradient is taken with respect to (k_x, k_y) . Note that your answer will depend on your choice of gauge.

The positive energy eigenstates are

$$|\mathbf{k}+\rangle = \begin{pmatrix} \cos \theta_{\mathbf{k}}/2 \\ \sin \theta_{\mathbf{k}}/2 e^{i\phi_{\mathbf{k}}} \end{pmatrix}$$

where $\cos \theta_{\mathbf{k}} = \Delta/E_{\mathbf{k}}$, $e^{i\phi_{\mathbf{k}}} = (k_x + ik_y)/|\mathbf{k}|$. The eigenket depends on \mathbf{k} through $\theta_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ which can be interpreted as coordinates on a Bloch sphere. Differentiating we obtain

$$\nabla_{\mathbf{k}} |\mathbf{k}+\rangle = \frac{1}{2} (\nabla_{\mathbf{k}} \theta_{\mathbf{k}}) \begin{pmatrix} -\sin \theta_{\mathbf{k}}/2 \\ \cos \theta_{\mathbf{k}}/2 e^{i\phi_{\mathbf{k}}} \end{pmatrix} + \begin{pmatrix} 0 \\ \sin \theta_{\mathbf{k}}/2 \nabla_{\mathbf{k}} e^{i\phi_{\mathbf{k}}} \end{pmatrix},$$

but only the second term enters in the Berry connection; the first term is orthogonal to $|\mathbf{k}+\rangle$:

$$\begin{aligned} \mathbf{A} &= i \langle \mathbf{k}+ | \nabla_{\mathbf{k}} | \mathbf{k}+ \rangle = \frac{i}{2} (\nabla_{\mathbf{k}} \theta_{\mathbf{k}}) (-\cos \theta_{\mathbf{k}}/2 \sin \theta_{\mathbf{k}}/2 + \sin \theta_{\mathbf{k}}/2 \cos \theta_{\mathbf{k}}/2) - \nabla_{\mathbf{k}} \phi_{\mathbf{k}} \sin^2 \theta_{\mathbf{k}}/2 \\ &= -\nabla_{\mathbf{k}} \phi_{\mathbf{k}} \sin^2 \theta_{\mathbf{k}}/2 \end{aligned}$$

One could do out the derivatives, but we will leave it as is because the above form will be more useful, and the Berry connection isn't particularly meaningful on its own anyway.

- (b) Compute the Berry phase by integrating the Berry connection around a circle of constant $|\mathbf{k}| = k_F$; $\gamma = \oint_{|\mathbf{k}|=k_F} \mathbf{A}(\mathbf{k}) \cdot d\mathbf{k}$. Please feel free to use mathematica, but show your work.

Integrating \mathbf{k} around a circle of constant $|\mathbf{k}| = k_F$ implies that the magnitude of $|\mathbf{k}|$ is fixed. We can therefore replace

$$\sin^2 \frac{\theta_{\mathbf{k}}}{2} \rightarrow \sin^2 \frac{\theta_{k_F}}{2} = \frac{1}{2}(1 - \cos \theta_{k_F})$$

where the subscript k_F just indicates that for the purposes of this problem θ only depends on the fixed value $k_F = |\mathbf{k}|$ not \mathbf{k} . The integral is

$$\oint_{|\mathbf{k}=k_F} \mathbf{A}_{\mathbf{k}} \cdot d\mathbf{k} = \frac{1}{2} (\cos \theta_{k_F} - 1) \oint \nabla_{\mathbf{k}} \phi_{\mathbf{k}} \cdot d\mathbf{k} = \frac{1}{2} (\cos \theta_{k_F} - 1) \int_0^{2\pi} d\phi_{\mathbf{k}} = \pi (\cos \theta_{k_F} - 1).$$

- (c) Compute the Berry curvature $\Omega(\mathbf{k}) = \nabla \times \mathbf{A}(\mathbf{k})$. Compute the Berry phase again by performing the associated surface integral and show that your answer agrees with the previous part. Again feel free to use mathematica, but show your work.

Using the product rule for curl we have

$$\Omega(\mathbf{k}) = \nabla \times \mathbf{A}(\mathbf{k}) = -\frac{1}{2} (1 - \cos \theta_{\mathbf{k}}) \nabla \times \nabla \phi_{\mathbf{k}} - \frac{1}{2} \sin(\theta_{\mathbf{k}}) \nabla_{\mathbf{k}} \theta_{\mathbf{k}} \times \nabla_{\mathbf{k}} \phi_{\mathbf{k}}$$

Using mathematica the above can be simplified to

$$\Omega(\mathbf{k}) = -\frac{\Delta}{2(|\mathbf{k}|^2 + \Delta^2)^{3/2}}.$$

The Berry phase can be computed directly by integrating the above explicit form, either in mathematica or by transforming to polar coordinates. But another way to do the integral is by using that $\nabla_{\mathbf{k}} \theta_{\mathbf{k}} \times \nabla_{\mathbf{k}} \phi_{\mathbf{k}}$ is the Jacobian associated with transforming between (k_x, k_y) and $(\phi_{\mathbf{k}}, \theta_{\mathbf{k}})$. With this

observation we have

$$\gamma = \int_{|\mathbf{k}| < k_F} \Omega(\mathbf{k}) d^2\mathbf{k} = -\frac{1}{2} \int_{|\mathbf{k}| < k_F} \sin(\theta_{\mathbf{k}}) d\phi_{\mathbf{k}} d\theta_{\mathbf{k}} = \pi \int_1^{\cos \theta_{k_F}} d \cos \theta_{k_F} = \pi(\cos \theta_{k_F} - 1)$$

in agreement with the previous part.

- (d) Show that the Berry phases are opposite for the negative energy eigenstates (you don't have to do both methods above, one would suffice). Also note that the Berry phases flip upon $\Delta \rightarrow -\Delta$.

The negative energy eigenstates are

$$|\mathbf{k}-\rangle = \begin{pmatrix} \sin \theta_{\mathbf{k}}/2 \\ -\cos \theta_{\mathbf{k}}/2 e^{i\phi_{\mathbf{k}}} \end{pmatrix}$$

The Berry connection for the negative states is

$$A_{\mathbf{k}}^- = -\cos^2 \theta_{\mathbf{k}}/2 \nabla_{\mathbf{k}} \phi_{\mathbf{k}} = (\sin^2 \theta_{\mathbf{k}}/2 - 1) \nabla_{\mathbf{k}} \phi_{\mathbf{k}} = -A_{\mathbf{k}}^+ - \nabla_{\mathbf{k}} \phi_{\mathbf{k}}$$

where the part proportional to the gradient of $\theta_{\mathbf{k}}$ vanishes in the same way as before. The connections are the opposite up to the second term, but the Berry phase associated to the second term always yields 2π which is equivalent to zero for a Berry phase.

Under $\Delta \rightarrow -\Delta$ we have $\cos \theta_{k_F} \rightarrow -\cos(\theta_{k_F})$ so the Berry phase

$$\gamma = \pi(\cos(\theta_{k_F}) - 1) \rightarrow \pi(-1 - \cos(\theta_{k_F})) = 2\pi - \pi(\cos(\theta_{k_F}) - 1) = 2\pi - \gamma$$

as claimed.