

Phys 251A Problem Set 7
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Throughout this problem set we will use the angular momentum algebra

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k, \quad [J_z, J_{\pm}] = [J_z, J_x \pm iJ_y] = \pm\hbar J_{\pm}, \quad J^2 = \sum_i J_i^2$$

where we will often work with sectors of fixed $J^2 = \hbar^2 j(j+1)$ because $[J^2, J_i] = 0$.

1. Beyond spin 1/2

- (a) Consider the angular momenta J_i for a spin 1 particle. The usual Hilbert space basis is given by the eigenstates of J_z , i.e. $|+\rangle = |j=1, m=1\rangle$ as well as $|0\rangle = |1, 0\rangle$ and $|-\rangle = |1, -1\rangle$. In this basis, using the angular momentum algebra, derive the explicit 3×3 matrices for $J_{x,y,z}$. These matrices generalize the matrices $\frac{\hbar}{2}\sigma_i$ that we used for spin 1/2.

Since J_z is diagonal and its action is $J_z |m\rangle = \hbar m |m\rangle$ we have

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We now use $J_{\pm} |m\rangle = \hbar c_{\pm}(m) |m \pm 1\rangle$ where $c_{\pm}(m) = \sqrt{j(j+1) - m(m \pm 1)}$. Here $j = 1$ so that $c_+(0) = \sqrt{2}$ and $c_+(-1) = \sqrt{2}$ which determine the nonzero matrix elements $\langle + | J_+ | 0 \rangle = \sqrt{2}\hbar$ and $\langle 0 | J_+ | -1 \rangle = \sqrt{2}\hbar$ respectively. We therefore have

$$J_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where we can obtain J_- either through the same technique or by taking the Hermitian conjugate of J_+ . We now can use $J_x = \frac{1}{2}(J_+ + J_-)$ and $J_y = -\frac{i}{2}(J_+ - J_-)$ to obtain

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

- (b) For spin 1/2 particles we observed that the Pauli matrices further satisfied $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. For spin 1, are the anticommutators also proportional to the identity?

No, it is easy to see that

$$J_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is not proportional to the identity matrix whereas $\sigma_z^2 = \frac{1}{2}\{\sigma_z, \sigma_z\} = 1$ is proportional to the identity.

- (c) Show that, in the state $|j, m\rangle$, now for a general j , we have

$$\langle J_x \rangle = \langle J_y \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{1}{2}\hbar^2(l(l+1) - m^2)$$

Using the expressions for J_x and J_y in terms of J_{\pm} we have

$$\begin{aligned} J_x |j, m\rangle &= \frac{1}{2}(J_+ + J_-) |j, m\rangle = \frac{1}{2}\hbar(c_+(j, m) |j, m+1\rangle + c_-(j, m) |j, m-1\rangle) \\ J_y |j, m\rangle &= -i\frac{1}{2}(J_+ - J_-) |j, m\rangle = -i\frac{1}{2}\hbar(c_+(j, m) |j, m+1\rangle - c_-(j, m) |j, m-1\rangle) \end{aligned}$$

so that $\langle J_x \rangle = \langle J_y \rangle = 0$. We can now calculate

$$\begin{aligned} \langle J_x^2 \rangle &= (\langle j, m | J_x)(J_x |j, m\rangle) \\ &= \frac{1}{4}\hbar^2 (c_+(j, m) \langle j, m+1 | + c_-(j, m) \langle j, m-1 |) (c_+(j, m) |j, m+1\rangle + c_-(j, m) |j, m-1\rangle) \\ &= \frac{1}{4}\hbar^2 (c_+(j, m)^2 + c_-(j, m)^2) \\ \langle J_y^2 \rangle &= (\langle j, m | J_y)(J_y |j, m\rangle) \\ &= \frac{1}{4}\hbar^2 (c_+(j, m) \langle j, m+1 | - c_-(j, m) \langle j, m-1 |) (c_+(j, m) |j, m+1\rangle - c_-(j, m) |j, m-1\rangle) \\ &= \frac{1}{4}\hbar^2 (c_+(j, m)^2 + c_-(j, m)^2) \end{aligned}$$

where we used that $c_{\pm}(j, m)$ are real and that J_x and J_y are Hermitian. We now use $c_{\pm}(j, m)^2 = j(j+1) + m(m \pm 1)$ to get

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{1}{4}(c_+(j, m)^2 + c_-(j, m)^2) = \frac{1}{2}\hbar^2(j(j+1) - m^2)$$

2. Consider the hydrogen atom Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}.$$

We will write it as

$$H = \gamma + \frac{1}{2m} \sum_{k=1}^3 \left(\hat{p}_k + i\beta \frac{\hat{x}_k}{r} \right) \left(\hat{p}_k - i\beta \frac{\hat{x}_k}{r} \right),$$

where \hat{p}_k and \hat{x}_k are, respectively, the Cartesian components of the momentum and position operators, and β and γ are real constants to be adjusted so that the two Hamiltonians are the same.

(a) Calculate the constants β and γ . Express them in terms of e^2 the Bohr radius a_0 and other constants.

$$\begin{aligned} H &= \gamma + \frac{\vec{p}^2}{2m} + \frac{i\beta}{2m} \sum_{k=1}^3 \left[\frac{x_k}{r}, p_k \right] + \frac{\beta^2}{2m} \\ &= \gamma + \frac{\beta^2}{2m} + \frac{\vec{p}^2}{2m} + \frac{i\beta}{2m} \sum_{k=1}^3 i\hbar \left(\frac{1}{r} - \frac{x_k^2}{r^3} \right) \\ &= \gamma + \frac{\beta^2}{2m} + \frac{\vec{p}^2}{2m} - \frac{\hbar}{mr} \beta \end{aligned}$$

Compare the Hamiltonian, we get

$$\frac{\hbar}{mr} \beta = \frac{e^2}{r}, \quad \gamma + \frac{\beta^2}{2m} = 0$$

which gives

$$\beta = \frac{me^2}{\hbar}, \quad \gamma = -\frac{me^4}{2\hbar^2}$$

quoting the Bohr radius (in Gaussian unit), $a_0 = \frac{\hbar^2}{me^2}$, we have,

$$\beta = \hbar/a_0, \quad \gamma = -\frac{e^2}{2a_0}$$

- (b) Explain why for any state $\langle H \rangle \geq \gamma$. Find the wavefunction of the state for which this energy inequality is saturated. This is the ground state of Hydrogen. Give the normalized wavefunction.

Write down the “annihilation” operator $\hat{A}_k = \hat{p}_k - i\beta \frac{\hat{x}_k}{r}$, so as the “creation” operator $\hat{A}_k^\dagger = \hat{p}_k + i\beta \frac{\hat{x}_k}{r}$. Then the expected value of Hamiltonian for any general state $|\psi\rangle$ is

$$\langle \psi | H | \psi \rangle = \gamma + \sum_{k=1}^3 \langle \psi | \hat{A}_k^\dagger \hat{A}_k | \psi \rangle = \gamma + \sum_{k=1}^3 \langle \hat{A}_k \psi | \hat{A}_k \psi \rangle \geq \gamma$$

The inequality is saturated when $\hat{A}_k |\psi\rangle = 0$ is a null state.

$$0 = \left(-i\hbar \frac{\partial}{\partial x_k} - i\hbar \frac{x_k}{a_0 r} \right) \psi(\vec{r})$$

$\psi(\vec{r}) = Ce^{-r/a_0}$ is a solution. Normalize it:

$$1 = \int dr^3 \psi^*(\vec{r}) \psi(\vec{r}) = 4\pi |C|^2 \int_0^\infty dr e^{-2r/a_0} r^2 = \pi |C|^2 a_0^3$$

Choose $C = \frac{1}{\sqrt{\pi a_0^3}}$, the normalized wavefunction is

$$\psi(\vec{r}) = \pi^{-1/2} a_0^{-3/2} e^{-r/a_0}$$

where $a_0 = \frac{\hbar^2}{me^2}$

3. Suppose you have a set of angular momentum operators $\hat{J}_i, i = 1, 2, 3$ that define an angular momentum $\hat{\mathbf{J}}$. A set of operators \hat{W}_i , with $i = 1, 2, 3$ is said to form a vector operator $\hat{\mathbf{W}}$ if

$$[\hat{J}_i, \hat{W}_j] = i\hbar \epsilon_{ijk} \hat{W}_k.$$

Note that $\hat{\mathbf{J}}$ itself is a vector operator.

- (a) Show that when $\hat{\mathbf{J}}$ is taken to be the orbital angular momentum $\hat{\mathbf{L}}$ the position operator $\hat{\mathbf{x}}$ is a vector operator.

$$[L_i, x_j] = [x_l p_k \epsilon_{ilk}, x_j] = \epsilon_{ilk} x_l [p_k, x_j] = -i\hbar \delta_{kj} \epsilon_{ilk} x_l = i\hbar \epsilon_{ijl} x_l$$

- (b) Show that if $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$ are vector operators, so is the cross product $\hat{\mathbf{U}} \times \hat{\mathbf{V}}$.

\hat{U} and \hat{V} are vector operators, meaning

$$[J_i, U_j] = i\hbar\epsilon_{ijk}U_k, \quad [J_i, V_j] = i\hbar\epsilon_{ijk}V_k$$

For the cross product,

$$\begin{aligned} [J_i, (\vec{U} \times \vec{V})_j] &= [J_i, \epsilon_{jkl}U_kV_l] = \epsilon_{jkl}[J_i, U_k]V_l + \epsilon_{jkl}U_k[J_i, V_l] \\ &= i\hbar(\epsilon_{jkl}\epsilon_{ikm}U_mV_l + \epsilon_{jkl}\epsilon_{ilm}U_kV_m) \\ &= i\hbar((\delta_{ij}\delta_{lm} - \delta_{jm}\delta_{il})U_mV_l + (\delta_{jm}\delta_{ik} - \delta_{ij}\delta_{km})U_kV_m) \\ &= i\hbar(\delta_{ij}\vec{U} \cdot \vec{V} - U_jV_i + U_iV_j - \delta_{ij}\vec{U} \cdot \vec{V}) \\ &= i\hbar(U_iV_j - U_jV_i) = i\hbar\epsilon_{ijk}(\vec{U} \times \vec{V})_k \end{aligned}$$

(c) Show that if $\hat{\mathbf{W}}$ is a vector operator then

$$[\hat{\mathbf{J}}^2, \hat{\mathbf{W}}] = 2i\hbar(\hat{\mathbf{W}} \times \hat{\mathbf{J}} - i\hbar\hat{\mathbf{W}})$$

Check that this formula holds when we choose $\hat{\mathbf{W}} = \hat{\mathbf{J}}$.

$$\begin{aligned} [\vec{J}^2, W_k] &= \delta_{ij}[J_iJ_j, W_k] = \delta_{ij}J_i[J_j, W_k] + \delta_{ij}[J_i, W_k]J_j = i\hbar\delta_{ij}(J_iW_l\epsilon_{jkl} + W_lJ_j\epsilon_{ikl}) \\ &= i\hbar(J_iW_l + W_lJ_i)\epsilon_{ikl} = i\hbar(2W_lJ_i\epsilon_{ikl} + i\hbar\epsilon_{ikl}\epsilon_{ilm}W_m) \\ &= i\hbar(2(\vec{W} \times \vec{J})_k - i\hbar 2\delta_{km}W_m) = 2i\hbar(\vec{W} \times \vec{J} - i\hbar\vec{W})_k \end{aligned}$$

when we choose $\vec{W} = \vec{J}$, the formula,

$$\text{LHS} = [\vec{J}^2, \vec{J}] = 0$$

while

$$(\vec{J} \times \vec{J})_k = J_iJ_j\epsilon_{ijk} = \frac{1}{2}\epsilon_{ijk}[J_i, J_j] = \frac{1}{2}i\hbar\epsilon_{ijl}J_l = \frac{1}{2}i\hbar 2\delta_{kl}J_l = i\hbar J_k$$

so

$$\text{RHS} = 2i\hbar(\vec{J} \times \vec{J} - i\hbar\vec{J}) = 0$$

4. Consider a particle moving in the field of a magnetic monopole. We recall the commutation relations of the dynamical momentum, $\boldsymbol{\pi} = -i\hbar\nabla - eA(\mathbf{r})$, and the magnetic field of the magnetic monopole

$$[\pi_i, \pi_j] = ie\hbar\epsilon_{ijk}B_k(\mathbf{r}), \quad B(\mathbf{r}) = \frac{g\mathbf{r}}{4\pi r^3}$$

where now $B(\mathbf{r})$ and r should be understood as functions of the vector of position operators \mathbf{r} . Since the monopole field is spherically symmetric, the angular momentum of a charge moving in a monopole field should be conserved. However the naive choice $\mathbf{L}_0 = \mathbf{r} \times \boldsymbol{\pi}$ does not work (classically or quantum mechanically); it does not satisfy the angular momentum algebra and it does not commute with the Hamiltonian. There should be some generator of rotations, on physical grounds, so let's obtain it by guessing.

(a) Find $f(r)$ such that

$$[L_i, H] = 0, \quad H = \frac{1}{2m}\boldsymbol{\pi}^2, \quad \mathbf{L} = \mathbf{L}_0 + f(r)\hat{\mathbf{r}}.$$

It is possible to show that this same choice leads to L_i satisfying the angular momentum algebra, but you do not need to do this here.

First we calculate

$$\begin{aligned}
2m[H, L_{0i}] &= [\pi_j \pi_j, \varepsilon_{ilm} r_l \pi_m] = \{\pi_j, \varepsilon_{ilm} [\pi_j, r_l \pi_m]\} \\
&= \{\pi_j, \varepsilon_{ilm} (-i\hbar \delta_{jl} \pi_m + i\hbar \varepsilon_{jmk} r_l B_k)\} \\
&= i\hbar e \{\pi_j, \varepsilon_{ilm} \varepsilon_{jmk} r_l B_k\} \\
&= -i\hbar e (\delta_{ij} \delta_{lk} - \delta_{ik} \delta_{lj}) \{\pi_j, r_l B_k\} \\
&= -i\hbar e \{\pi_j, \delta_{ij} r_l B_l - r_j B_i\} \\
&= \frac{-i\hbar e g}{4\pi} \left\{ \pi_j, \frac{1}{r} \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \right\},
\end{aligned}$$

where in going from the second to the third line we used $\varepsilon_{ijm} \{\pi_j, \pi_m\} = 0$ because the anticommutator is symmetric in relabeling $j \leftrightarrow m$ while ε_{ijm} is antisymmetric.

As claimed the naive angular momentum doesn't commute. We now compute the commutator of the proposed correction. It is simpler to define $g(r) = f(r)/r$, such that $f(r)\hat{\mathbf{r}} = g(r)\mathbf{r}$. Using that π_j also acts as $-i\hbar$ times a derivative with respect to r_j when commuting functions of r_j (the part by which it differs from p_j commutes with r_j) we have

$$2m[H, g(r)r_i] = [\pi_j \pi_j, g(r)r_i] = \{\pi_j, [\pi_j, g(r)r_i]\} = -i\hbar \{\pi_j, g(r)\delta_{ij} + \partial_{r_j} g(r)r_i\}.$$

We see that if we choose $g(r) = c/r$ for some constant c (equal to $f(r)$) we have

$$g(r)\delta_{ij} + \partial_{r_j} g(r)r_i = \frac{c}{r} \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right)$$

such that $[L_i, H] = [L_{0i} + g(r)r_i, H] = 0$ if $c = -eg/4\pi$.

- (b) Unlike ordinary orbital angular momentum, we have found that \mathbf{L} has a fixed component in the $\hat{\mathbf{r}}$ direction, proportional to the f that you found, that is not generated by the motion of the particle, that we are forced to consider. Compare this component to the Dirac quantization condition that we derived in lecture, $eg = 2\pi\hbar n$, and conclude that Dirac quantization is equivalent to angular momentum quantization. You should also conclude that the minimal monopole charge pair allowed by Dirac quantization has *half*-integer orbital angular momentum.

We found that we had to add another piece of orbital angular momentum $-\frac{eg}{4\pi}\hat{\mathbf{r}}$ that is not associated with motion of the particle, and is orthogonal to the other components (it points radially outward, unlike typical orbital angular momentum $\mathbf{r} \times \mathbf{p}$ which is always orthogonal to $\hat{\mathbf{r}}$). We have, by Dirac quantization,

$$-\frac{eg}{4\pi}\hat{\mathbf{r}} = -\frac{1}{2}\hbar n \hat{\mathbf{r}}$$

such that the radial part of angular momentum is quantized to be integer or half integer if and only if Dirac quantization is satisfied. Furthermore, for odd n , we see that this piece of orbital angular momentum is quantized to be a *half*-integer. With some more work it is possible to show that $L^2 = \hbar^2 l(l+1)$ where l is a half integer if and only if $n/2$ is.