

**Phys 251A Problem Set 8**  
**Date posted: November 11, 2023**  
**Due date: November 27, 2023**

1. True/False. No need to explain, just write “T” or “F.”

(a) The trace of angular momentum operators,  $\text{Tr } J_i$ , always vanishes

**True** We have showed that  $J_z$  has eigenvalues from  $+\hbar j$  through  $-\hbar j$ , symmetrically around zero. So the sum of eigenvalues are zero. Alternatively, one can take the trace of the angular momentum algebra  $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$  and use that the trace of a commutator is zero.

(b) If  $J_{1i}$  and  $J_{2i}$  are angular momentum operators for two distinct particles “1” and “2” then the relative angular momenta,  $J_{-i} = J_{1i} - J_{2i}$ , also satisfies the angular momentum algebra.

**False** We have

$$[J_{1i} - J_{2i}, J_{1j} - J_{2j}] = [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}] = i\hbar\varepsilon_{ijk}(J_{1k} + J_{2k})$$

whereas if these operators satisfied the angular momentum algebra we would have  $J_{1k} - J_{2k}$  on the right hand side instead.

(c) If  $[H, L_x] = 0$  and  $[H, L_y] = 0$ , where  $L_x$  and  $L_y$  are the orbital angular momenta in the  $x$  and  $y$  directions, then the Hamiltonian is rotationally symmetric in all directions.

**True** We have  $[H, L_x L_y] = [H, L_y L_x] = 0$  since  $H$  commutes with each operator individually. Then we must have  $[H, L_z] = -\frac{i}{\hbar}[H, [L_x, L_y]] = 0$  so that  $H$  commutes with all angular momentum operators and is therefore rotationally symmetric.

(d) There is a unitary operator  $R_j$  such that  $R_j^\dagger J_i R_j = J_i$  for  $i \neq j$  but  $R_j^\dagger J_j R_j = -J_j$ , where  $J_i$  are the angular momentum operators.

**False** Acting on the commutation relation we have

$$i\hbar\varepsilon_{ijk}R_k^\dagger J_k R_k = -i\hbar\varepsilon_{ijk}J_k$$

But the right hand side is

$$R_k^\dagger [J_i, J_j] R_k = [R_k^\dagger J_i R_k, R_k^\dagger J_j R_k] = [J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$$

since the epsilon symbol ensures that neither  $i$  or  $j$  is equal to  $k$ .

2. Consider three spin one particles, labeled 1,2, and 3, with spin operators  $\mathbf{S}_1$ ,  $\mathbf{S}_2$  and  $\mathbf{S}_3$ . The spins are placed on a circle and the interactions are between nearest neighbors. The Hamiltonian is

$$H = -\frac{\Delta}{\hbar^2}(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1). \quad (1)$$

You will find the total spin operator  $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$  useful, as well as the angular momentum decomposition  $j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \dots \oplus |j_1 - j_2|$ .

(a) What is the dimensionality of the space of the three combined particles?

The dimension of each particle's Hilbert space is 3, and Hilbert space dimension multiplies under combining (tensor producting), so that we have  $3 * 3 * 3 = 27$ .

- (b) Rewrite the Hamiltonian by expanding out  $S^2 = \mathbf{S} \cdot \mathbf{S}$ .

We have

$$S^2 = S_1^2 + S_2^2 + S_3^2 + 2(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_1 \cdot \mathbf{S}_3)$$

so that

$$H = -\frac{\Delta}{2\hbar^2}(S^2 - S_1^2 - S_2^2 - S_3^2) = -\frac{\Delta}{2\hbar^2}S^2 + 3\Delta$$

where we used that  $S_i^2 = \hbar^2 1(1+1) = 2\hbar^2$ .

- (c) Find all the energy eigenvalues and their degeneracies

The energy eigenvalues are given by

$$E = \frac{\Delta}{2}(6 - j(j+1))$$

where  $j$  is the total spin;  $S^2 = \hbar^2 j(j+1)$ . To find the total spins and their degeneracies we compute

$$1 \otimes 1 \otimes 1 = (2 \oplus 1 \oplus 0) \otimes 1 = (2 \otimes 1) \oplus (1 \otimes 1) \oplus (0 \otimes 1) = 3 \oplus 2 \oplus 1 \oplus 2 \oplus 1 \oplus 0 \oplus 1$$

So that there are  $2 * 3 + 1 = 7$  states with  $j = 3$  and  $E = \frac{\Delta}{2}(6 - 12) = -3\Delta$  as the ground state. The next lowest energy consists of two spin two representations, consisting of  $2 * (2 * 2 + 1) = 10$  states with  $E = \frac{\Delta}{2}(6 - 6) = 0$ . There are three representations with  $j = 1$ , or  $3 * (2 * 1 + 1) = 9$  states and  $E = \frac{\Delta}{2}(6 - 2) = 2\Delta$ , and finally one representation with  $j = 0$ , one state, and  $E = 3\Delta$ . The total number of states sum as  $7 + 10 + 9 + 1 = 27$  as expected.

- (d) Write down the ground state of maximal  $S_z$ , in the form  $|m_1, m_2, m_3\rangle$ , as well as the ground state of next lowest  $S_z$  explicitly. Explain how you would find the other ground states as well, though for these a schematic form is sufficient.

The ground states correspond to  $j = 3$  and the state of maximal  $S_z$  is  $|+++ \rangle$ . Acting with the lowering operator for the total spin then gives  $J_- |+++ \rangle = \hbar\sqrt{2}(|0++ \rangle + |+0+ \rangle + |++0 \rangle)$ . as the state of next lowest  $S_z$  (up to normalization). The 5 other ground states can be obtained by acting with  $J_-$  five more times.

3. Adding angular momenta (notational advice: if you find yourself getting confused labeling different types of states all with just explicit numbers, it might help to use  $\pm, 0$  for spin 1 values of  $m$ , and similarly  $\pm$  or  $\uparrow\downarrow$  for spin  $1/2$  values of  $m$ , while reserving numbers for the total angular momentum states  $|j, m\rangle$ .)

- (a) Imagine a particle has orbital angular momentum  $l = 1$  and spin  $s = \frac{1}{2}$ ; what are the six states  $|j, m\rangle$  in terms of the basis states  $|m_l, m_s\rangle$ ? You can check your answer against Clebsch-Gordan coefficients looked up online, but you should derive this case yourself and show your work on your submission.

When adding a spin one representation to a spin half representation, the two possibilities are  $j = 3/2$  and  $j = 1/2$  (there is no other half integer in between, and these are the largest and smallest possible values for the total  $j$  respectively). There are four states in the  $j = 3/2$  representation and two states in the  $j = 1/2$  representation, which matches with the  $2 * 3 = 6$  states expected from combining a spin half and spin one representation. Our goal is to enumerate these six states.

Let us begin with the  $j = 3/2$  representation, with states labeled as  $|\frac{3}{2}, m\rangle$  on the left hand side of equations below, and superpositions of the states  $|m, s\rangle$  on the right hand side, where  $m = +, 0, -$  and  $s = \uparrow, \downarrow$  for clarity. There is only one possibility for the highest weight state

$$|\frac{3}{2}, \frac{3}{2}\rangle = |+, \uparrow\rangle.$$

Acting with  $\hbar^{-1}J_-$  we obtain, up to normalization

$$|\frac{3}{2}, \frac{1}{2}\rangle = \hbar^{-1}J_- |+, \uparrow\rangle = \sqrt{2}|0, \uparrow\rangle + |+, \downarrow\rangle$$

where we used the appropriate factors of  $c_{\pm}(j, m)$  when acting with  $J_- = L_- + S_-$  on the spin one and spin half particles. We could act with the lowering operator again but here its easier to just look at the lowest weight state and apply the raising operator. Completely analogously we have

$$|\frac{3}{2}, -\frac{3}{2}\rangle = |-, \downarrow\rangle, \quad |\frac{3}{2}, -\frac{1}{2}\rangle = \hbar^{-1}J_+ |-, \downarrow\rangle = \sqrt{2}|0, \downarrow\rangle + |-, \uparrow\rangle.$$

Now we look for the  $j = 1/2$  states. The highest weight state must be annihilated by  $J_+$ . With the general ansatz of correct  $J_z$ , we calculate  $J_+(a|+, \downarrow\rangle + b|0, \uparrow\rangle) = \hbar(a + \sqrt{2}b)|+, \uparrow\rangle$ . We therefore choose  $b = -1$  and  $a = \sqrt{2}$  up to normalization so that

$$|\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{2}|+, \downarrow\rangle - |0, \uparrow\rangle.$$

Lowering an ansatz for the lowest weight state analogously yields an expression for the lowest weight state. Alternatively use  $J_-$  applied to the above. Either way one obtains

$$|\frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{2}|-, \uparrow\rangle - |0, \downarrow\rangle.$$

We caution that we have not normalized the above states. If they are combined further with other states with relative signs the relative normalization can matter. This will not be the case in the next part however.

- (b) Consider three spin 1/2 particles with basis states  $|m_1, m_2, m_3\rangle$ . Using the states we derived in lecture for adding two spin 1/2 particles, together with the previous part that adds a spin 1 particle to a spin half particle, find the possible values for the total angular momentum  $j$  and derive the eight states  $|j, m\rangle$  for the entire system (you should not have to act with any operators, perhaps besides  $J_z$  where the actions is very straightforward, in this sub part.)

It is useful to schematically write this decomposition using tensor product and direct sum notation. We denote the vector space associated with a spin  $j$  as  $V_j$ . Then we decompose three spin 1/2 particles as follows:

$$V_{1/2} \otimes V_{1/2} \otimes V_{1/2} = (V_1 \oplus V_0) \otimes V_{1/2} = (V_1 \otimes V_{1/2}) \oplus (V_0 \otimes V_{1/2}) = (V_{3/2} \oplus V_{1/2}) \oplus V_{1/2}. \quad (2)$$

Here,  $\otimes$  denotes the tensor product and  $\oplus$  denotes the direct sum. In lecture we derived the states associated with  $V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0$ . The states on the right, labeled as  $|j, m\rangle$ , are

$$\begin{aligned} |1, 1\rangle &= |\uparrow\uparrow\rangle \\ |1, 0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1, -1\rangle &= |\downarrow\downarrow\rangle \\ |0, 0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{aligned}$$

We have normalized the above states since we will soon use them in combination with the previous part. Their phase is also consistent with the typical convention of real  $c_{\pm}$ . We similarly wrote the states associated with  $V_1 \otimes V_{1/2} = V_{3/2} \oplus V_{1/2}$  above. The states for the  $V_{3/2}$  representation come from combining the  $j = 1$  states above, which we will call  $|+\rangle, |0\rangle, |-\rangle$  as before, with the third spin  $1/2$  (see (2) with the coefficients given by the previous part:

$$\begin{aligned} \left|\frac{3}{2}, \frac{3}{2}\right\rangle &= |+\uparrow\rangle = |\uparrow\uparrow\uparrow\rangle \\ \left|\frac{3}{2}, \frac{1}{2}\right\rangle &= |+\downarrow\rangle + \sqrt{2}|0, \uparrow\rangle = |\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle \\ \left|\frac{3}{2}, -\frac{1}{2}\right\rangle &= |-\uparrow\rangle + \sqrt{2}|0, \downarrow\rangle = |\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle \\ \left|\frac{3}{2}, -\frac{3}{2}\right\rangle &= |-\rangle \otimes |\downarrow\rangle = |\downarrow\downarrow\downarrow\rangle \end{aligned}$$

The first  $V_{1/2}$  also has the same origin. Since there are two  $V_{1/2}$  representations in (2) we will label the kets with “1” and “2”.

$$\begin{aligned} \left|\frac{1}{2}, \frac{1}{2}\right\rangle_1 &= (\sqrt{2}|+\downarrow\rangle - |0, \uparrow\rangle) = \sqrt{2}|\uparrow\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_1 &= \sqrt{2}|-\uparrow\rangle - |0, \downarrow\rangle = \sqrt{2}|\downarrow\downarrow\uparrow\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle) \end{aligned}$$

Finally, the second  $V_{1/2}$  in (2) comes from  $V_0 \otimes V_{1/2} = V_{1/2}$  and therefore corresponds to the states

$$\begin{aligned} \left|\frac{1}{2}, \frac{1}{2}\right\rangle_2 &= |0\uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_2 &= |0\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle) \end{aligned}$$