Phys 251A Problem Set 9
Date posted: November 27, 2023
Due date: December 5, 2023

1. Consider a perturbed Harmonic oscillator Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{g}{l^3}x^3$$

where $g \ll \omega$ has the dimensions of energy and $l = \sqrt{\hbar/m\omega}$ is the oscillator length. You should treat the term proportional to g perturbatively.

(a) Express the above problem in terms of creation and annihilation operators (this will make subsequent calculations easier). Show that to first order in g the ground state energy shift is zero. Calculate the shift to second order in g.

In terms of creation and annihilation operators the first two terms are $H_0 = \omega(a^\dagger a + \frac{1}{2})$, as usual. Using $x = \frac{l}{\sqrt{2}}(a + a^\dagger)$, the third term is $V = \frac{g}{2\sqrt{2}}(a + a^\dagger)^3$.
It will be useful to calculate

$$\begin{aligned}(a + a^\dagger)^3 |0\rangle &= (a^\dagger a^\dagger a^\dagger + a a^\dagger a^\dagger + a^\dagger a a^\dagger)|0\rangle \\ &= (\sqrt{6}|3\rangle + 2|1\rangle + |1\rangle) = \sqrt{6}|3\rangle + 3|1\rangle\end{aligned} \tag{1}$$

where we used that we could drop any term with an a on the right, as well as any term with more a's then $a^\dagger$'s. We also used $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$. There are no $|0\rangle$ states on the right hand side which implies that the first order shift in the ground state vanishes

$$E_0^{(1)} = \langle 0| V |0\rangle = \frac{g}{2\sqrt{2}}\langle 0|(a + a^\dagger)^3|0\rangle = 0.$$

The second order shift is

$$E_0^{(2)} = -\sum_{n>0}\frac{|\langle n| V |0\rangle|^2}{n\hbar\omega} = -\frac{g^2}{(2\sqrt{2})^2}\left(\frac{6}{3\hbar\omega} - \frac{9}{\hbar\omega}\right) = -\frac{11g^2}{8\hbar\omega},$$

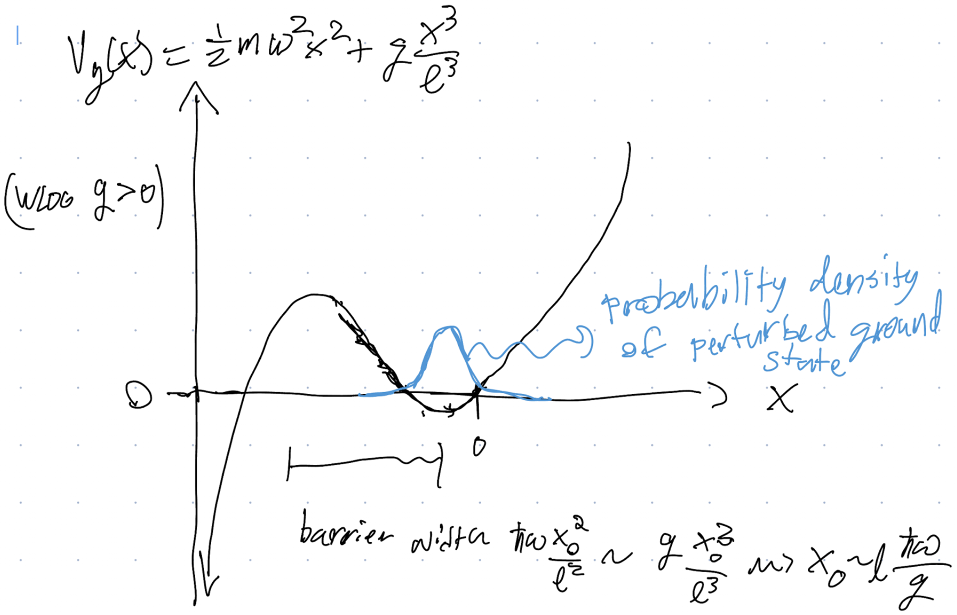
where we used that we can label the unperturbed states here by $|n\rangle$, where in a slight abuse of notation we mean the Harmonic oscillator eigenstates with $a^\dagger a|n\rangle = n|n\rangle$. We also used (1), and that the energy denominator is $E_0^{(0)} - E_n^{(0)} = -n\hbar\omega$.

(b) Calculate the perturbed wavefunction to leading order in g.

The perturbed wavefunction is

$$\begin{aligned}|0^{(1)}\rangle &= -\sum_{n\neq 0}\frac{\langle n| V |0\rangle}{n\hbar\omega}|n\rangle \\ &= -\frac{3g}{2\sqrt{2}\hbar\omega}|1\rangle - \frac{\sqrt{6}g}{2\sqrt{2}3\hbar\omega}|3\rangle \quad = -\frac{3g}{2\sqrt{2}\hbar\omega}|1\rangle - \frac{g}{2\sqrt{3}\hbar\omega}|3\rangle\end{aligned}$$

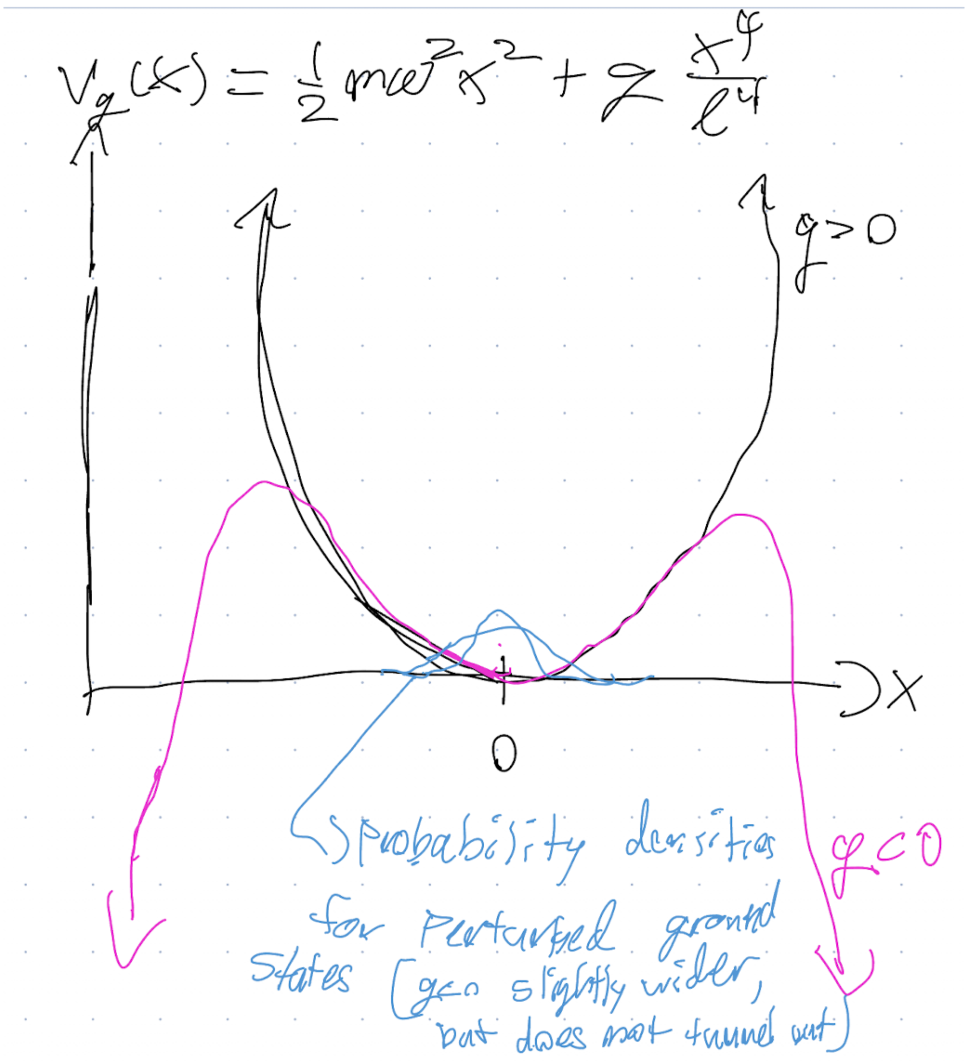
(c) Sketch the potential $V(x)$ in the above Hamiltonian for some small, nonzero g, and, on top of it, sketch the perturbed wavefunction to leading order in g. Does perturbation theory capture the true ground state? What physical process does perturbation theory not take into account? [If you prefer to type your problem set and don't want to make a figure, a sufficiently vivid description will suffice here]



The perturbation theory predicts a finite energy, with a wavefunction that is weakly perturbed away from the original non-perturbed one . However, the *global* minimum of the potential is at one of $x = -\infty$ (for $g > 0$) or $x = +\infty$ (for $g < 0$), and it is unbounded from below. By sliding towards the highly negative potential regions, the particle can lower its energy by an unbounded amount. It is kept from doing so by a potential barrier that gets wider and taller as $g \to 0$, such that a tunneling process is required. Perturbation theory is incapable of describing this tunneling process, which turns out to be related to the fact that there are certain real functions that look perfectly well behaved but are not equal to their Taylor series (consider e^{-1/x^2}, which has all derivatives equal to zero and thus cannot be equal to its Taylor series. A physical process that has this functional dependence, like tunneling turns out to, cannot be captured within perturbation theory.

(d) Let us consider the perturbation $\frac{g}{l^4}x^4$ instead. This term is typically present in symmetric potential wells, and leads to unequal spacing between energy levels ("anharmonic oscillator"). By sketching the total potential for small but nonzero $g > 0$ and $g < 0$, argue that a power series expansion of the ground state energy around $g = 0$ has zero radius of convergence.

Here we are using the fact that any power series $\sum_n c_n x^n$ *has a radius of convergence* $R \geq 0$ *such that the series converges for* $|x| < R$ *but diverges for* $|x| > R$. *Note that series with* $R = 0$ *are not useless, they can be regarded as asymptotic series. A good approximation can be obtained by keeping terms until they start to increase in magnitude. Sometimes advanced resummation techniques can be used to extract a convergent result (Borel summation)*



As we can see from the figure above, while for $g > 0$ the potential is well-behaved and is expected to confine the particle, the potential eventually turns negative for any $g < 0$. Since for $g < 0$ the potential can become arbitrarily negative as $x \to \infty$, the ground state energy must be $-\infty$; by moving further and further away we can construct variational states with arbitrarily negative energy, for example.

We therefore conclude that the ground state energy is finite for $g \geq 0$ and $-\infty$ for $g < 0$. Consider a power series expansion of the ground state energy, like one would try to derive in perturbation theory $E_{\mathrm{GS}} = \sum_n c_n g^n$. This series cannot converge for $g < 0$, which implies that the radius of convergence of this expansion is zero. Thus, the series does not converge for $g > 0$ either, and it should be interpreted as an asymptotic series for $g > 0$ rather than a convergent Taylor expansion.

2. A perturbed three dimensional harmonic oscilator: consider the Hamiltonian

$$H = \hbar\omega(N_1 + N_2 + N_3 + \frac{3}{2}) + \kappa L_3$$

where $N_i = a_i^\dagger a_i$ are the number operatorss of the i'th oscillator and $L_3 = x_1 p_2 - x_2 p_1$ is the z component of angular momentum. The unperturbed Hamiltonian, H_0, is H evaluated at $\kappa = 0$.

   (a) Let us label the unperturbed states as $|n_1, n_2, n_3\rangle$ where n_i is the eigenvalue of N_i. How many linearly independent states are there with energy $\frac{5}{2}\hbar\omega$ for $\kappa = 0$?

For the energy $\frac{5}{2}\hbar\omega$ we must have one of $N_i = 1$ and the others zero. There are thus three states, labeled by the N_i that is nonzero, i.e. $|100\rangle, |010\rangle, |001\rangle$.

(b) Derive an effective Hamiltonian, within the degenerate subspace you described in the previous part, up to first order in κ, and find its energy eigenvalues and eigenvectors.

The zeroth order effective Hamiltonian is simply a constant $\frac{5}{2}\hbar\omega$. Let us label the basis states $|100\rangle, |010\rangle, |001\rangle$ as $|1\rangle, |2\rangle, |3\rangle$ respectively corresponding to the oscillator that has been excited. We now write L_3 in terms of creation and annhiliation operators and obtain

$$L_3 = \frac{\hbar}{2i}\left((a_1 + a_1^\dagger)(a_2 - a_2^\dagger) - (a_2 + a_2^\dagger)(a_1 - a_1^\dagger)\right) = -i\hbar(a_1^\dagger a_2 - a_1 a_2^\dagger)$$

such that

$$\begin{aligned} L_3 |1\rangle &= -i\hbar(a_1^\dagger a_2 - a_1 a_2^\dagger)|100\rangle = i\hbar|010\rangle = i\hbar|2\rangle \\ L_3 |2\rangle &= -i\hbar(a_1^\dagger a_2 - a_1 a_2^\dagger)|010\rangle = i\hbar|010\rangle = -i\hbar|2\rangle \\ L_3 |3\rangle &= -i\hbar(a_1^\dagger a_2 - a_1 a_2^\dagger)|001\rangle = 0 \end{aligned}$$

so that, as a matrix in this three dimensional space, the first order correction is

$$h^{(1)}_{ll'} = \langle l|\kappa L_3|l'\rangle = \hbar\kappa\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenstates of the effective Hamiltonian are read off as (it is σ_y in the top left block):

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

with eigenvalue shifts $\hbar\kappa, -\hbar\kappa, 0$, relative to $\frac{5}{2}\hbar\omega$, respectively.

(c) Write the eigenstates of H, with energies near $\frac{5}{2}\hbar\omega$, in terms of linear combinations of $|n_1 n_2 n_3\rangle$ to first order in κ.

These can be read off from the eigenstates of the effective Hamiltonian (and confirmed by the action of L_3 as derived above). We will label the states as $|1, m\rangle$, with eigenvalues $\frac{5}{2}\hbar\omega + m\hbar\kappa$ where $m = \pm, 0$.

$$|1, \pm\rangle = \frac{1}{\sqrt{2}}(|100\rangle \pm i|010\rangle), \qquad |1, 0\rangle = |001\rangle.$$

(d) Are there corrections beyond first order in κ in this case? Why or why not? Hint: what is $[H_0, L_3]$?

No, it is straightforward to verify, using the action of L_3 derived in part (a) for example, that the states in the previous part are exact eigenstates of $H_0 + \kappa L_3$. This occurs because $[H_0, L_3] = 0$, so that the action of L_3 cannot change the eigenvalue of H. Therefore L_3 acts within the degenerate subspaces of H, and after diagonalizing L_3 within each degenerate subspace the problem has been solved exactly.