Phys 251A Problem Set 1
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1. In this problem we will prove some properties of the commutator $[A, B] = AB - BA$ and anticommutator $\{A, B\} = AB + BA$, where A, B are operators on some Hilbert space. The notation $\mathrm{ad}_A^{\pm}(B) = [A, B]_{\pm} = AB \mp BA$, such that $[A, B]_+$ is the commutator and $[A, B]_-$ is the anticommutator, can be useful.
   (a) Show that $[A, BC]_+ = [A, B]_{\pm}C \pm B[A, C]_{\pm}$. Convince yourself of the similarity to the "product rule" for derivatives: $\mathrm{ad}_A^{+}(BC) = \mathrm{ad}_A^{\pm}(B)C \pm B\,\mathrm{ad}_A^{\pm}(C)$.
   (b) Show, preferably using the previous identity, that
   $$[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B$$
2. Consider a matrix X, that is not necessarily Hermitian or Unitary, written as
   $$X = a_0 + \boldsymbol{a} \cdot \boldsymbol{\sigma}$$
   where $a_{0,1,2,3}$ are numbers, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices
   $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
   (a) Relate a_0 and a_k to $\mathrm{tr}\,X$ and $\mathrm{tr}(X\sigma_k)$.
   (b) Write a_0 and a_k in terms of the matrix elements X_{ij}.
3. Using bra-ket notation prove or evaluate the following
   (a) $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for operators X and Y
   (b) $(XY)^\dagger = Y^\dagger X^\dagger$ for operators X and Y.
   (c) $\exp(if(A))$ in ket-bra form in terms of the eigenvalues and eigenstates of A, for a function f that can be expanded in a power series if you wish.
   (d) $\sum_n \overline{\psi_n(x_1)}\psi_n(x_2)$ where n labels a complete set of states and $\psi_n(x) = \langle x|n\rangle$.
4. Suppose $|m\rangle$ and $|n\rangle$ are eigenstates of a Hermitian operator A. Find a general condition under which we can conclude that $|m\rangle + |n\rangle$ is an eigenstate of A.
5. Consider a Hermitian operator A with non-degenerate eigenvalues a_n labeled by n, and, using the eigenbasis of A,
   (a) Prove that $\prod_n (A - a_n) = 0$
   (b) Explain the action of $X_m = \prod_{n\neq m} \frac{A-a_n}{a_m-a_n}$.
   (c) Write out the previous two parts for $A = S^z = \frac{\hbar}{2}\sigma_z$, the spin operator in the z direction for a spin $\frac{1}{2}$ particle.
6. Consider a Hamiltonian of a two state system
   $$H = \varepsilon(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|), \tag{1}$$
   where ε has dimensions of energy. Find the energy eigenvalues and eigenvalues of H (in terms of $|1\rangle$ and $|2\rangle$).
7. Prove the Cauchy-Schwarz identity $|\langle a|b\rangle| \leq \|a\|\|b\|$ by observing
   $$\left(\langle a| + \overline{\lambda}\langle b|\right)\left(|a\rangle + \lambda|b\rangle\right) \geq 0, \tag{2}$$
   for all complex numbers λ, and then choosing λ appropriately.
8. Suppose two Hermitian operators anticommute: $\{A, B\} = 0$. Is it possible to have a simultaneous eigenstate of both A and B? Justify your answer

9. True or false questions. Explain your reasoning.
   (a) The commutator of two Hermitian operators is Hermitian
   (b) When two Hermitian operators A and B commute, any eigenvector of A is also an eigenvector of B.
   (c) An operator that is both Unitary and Hermitian must square to the identity.

BONUS : Let us suppose that the silver atoms in the Stern-Gerlach experiment have classical magnetic moments $\boldsymbol{\mu}$ that are random in direction; the direction is uniformly distributed over the sphere of all possible directions. Furthermore suppose their magnitude $\mu_0 = |\boldsymbol{\mu}|$ is drawn from a continuous probability distribution $p(\mu_0)$. Compute the distribution of μ_z values that would be seen in this case, in terms of the function $p(\mu_0)$.