Phys 251A Problem Set 9
Date posted: November 22, 2024
Due date: December 4, 2024

1. Consider a particle on a ring of radius R with kinetic energy

$$H_0 = -\frac{1}{2mR^2}\partial_\phi^2$$

where ϕ is the azimuthal angle and the wavefunction $\psi(\phi) = \psi(\phi + 2\pi)$ is periodic.

(a) Calculate the energy eigenvalues and eigenfunctions of H_0.

(b) Let us put the particle, which has charge q, in an electric field pointing in the x direction of strength ε, which leads to the full Hamiltonian

$$H = H_0 - q\varepsilon R\cos\phi.$$

Calculate the new ground state wavefunction to first order in ε.

(c) Use the new ground state wavefunction to calculate the induced electric dipole moment in the x direction

$$p_x = \frac{\langle\psi|\, qx\,|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\langle\psi|\, qR\cos\phi\,|\psi\rangle}{\langle\psi|\psi\rangle}$$

to first order in ε. Then use this to calculate the polarizability of the system

$$\alpha = \left.\frac{dp_x}{d\varepsilon}\right|_{\varepsilon=0}$$

2. *Upper and lower bounds on ground state energy.* You may find the following integrals useful.

$$\int_{-\infty}^{\infty} dx x^2 e^{-\frac{x^2}{d^2}} = d^3\frac{\sqrt{\pi}}{2}, \qquad \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{d^2}} = d\sqrt{\pi}$$

(a) Consider a Harmonic oscillator Hamiltonian

$$H = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2x^2$$

Bound the ground state energy from above with the variational principle (copied below) using a Gaussian trial state of variable width

$$E_0 \leq \frac{\langle\psi_d|\, H\,|\psi_d\rangle}{\langle\psi_d|\psi_d\rangle}, \qquad \psi_d(x) = e^{-\frac{x^2}{2d^2}}$$

Since d is arbitrary, optimize your bound by minimizing it over d. You will have an easier time if you use that $\langle x\rangle = \langle p\rangle = 0$.

(b) We will now lower bound the ground state energy. Using the Heisenberg uncertainty principle, show that the ground state energy is lower bounded as

$$E_0 \geq \frac{\hbar^2}{8m\langle x^2\rangle} + \frac{1}{2}m\omega^2\langle x^2\rangle$$

where $\langle x^2\rangle$ is the expectation value in the ground state of x^2. By computing the minimum of the function $f(\lambda) = \frac{\hbar^2}{8m\lambda} + \frac{1}{2}m\omega^2\lambda$, show that

$$E_0 \geq \frac{1}{2}\hbar\omega$$

and that this lower bound matches the upper bound of the previous part, pinning the ground state energy to the value $\frac{1}{2}\hbar\omega$.

(c) Now consider

$$H = -\frac{\hbar^2}{2m}\partial_x^2 + \lambda|x|$$

where λ has units of energy per unit length. Upper bound the ground state energy of H using a gaussian trial state, as you did in part (a), and optimize the value of d.