Phys 251A Problem Set 2

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1. Using that $|+z\rangle = |\uparrow\rangle$ and $|-z\rangle = |\downarrow\rangle$ are orthonormal, show that

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k, \qquad \{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij}$$

where

$$S_z = \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|), \qquad S_x = \frac{\hbar}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|), \qquad S_y = \frac{\hbar}{2}(-i\,|\uparrow\rangle\langle\downarrow| + i\,|\downarrow\rangle\langle\uparrow|).$$

2. Consider the spin operator $S_{\boldsymbol{n}} = \boldsymbol{n}\cdot\boldsymbol{S} = n_xS_x + n_yS_y + n_zS_z$ where $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector in an arbitrary direction characterized by a polar angle θ and azimuthal angle ϕ.
   (a) Construct the state $|\boldsymbol{n}+\rangle$, which is the eigenstate of $S_{\boldsymbol{n}}$ of positive eigenvalue, in terms of the $|\pm z\rangle$ basis states and the angles (θ, ϕ).
   (b) Compute $\langle(\Delta S_x)^2\rangle$ and $\langle(\Delta S_y)^2\rangle$ in the state $|\boldsymbol{n}+\rangle$ in terms of the angles (θ, ϕ). Convince yourself your answer makes sense by choosing some special (θ, ϕ).
   (c) Calculate $\langle[S_x, S_y]\rangle$ in the state $|\boldsymbol{n}+\rangle$. Show that the uncertainty relationship $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle \geq \frac{1}{4}|\langle[S_x, S_y]\rangle|^2$ is satisfied.
   (d) For which $\boldsymbol{n}$ is the uncertainty product $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle$ maximized?

3. Consider a momentum operator p that has eigenstates $|p\rangle$. Define an infinitesimal boost operator by the relationship $B(dp)\,|p_0\rangle = |p_0 + dp\rangle$. Write $B(dp) = 1 + iW\,dp$
   (a) Show that $B(dp)$ is unitary
   (b) Write $B(dp) = 1 + iW\,dp$. Show that W is Hermitian.
   (c) Calculate $[W, p]$, and write an ansatz of W in terms of x that satisfies this commutation relationship and thus generates the boost $B(dp)$.

4. Using the canonical commutation relationship $[x, p] = i\hbar$ and the identity proved in the last problem set, $\mathrm{ad}_A(BC) = \mathrm{ad}_A(B)C + B\,\mathrm{ad}_A(C)$ where $\mathrm{ad}_X(Y) = [X, Y]$,
   (a) Show that $[p, x^n] = -i\hbar n x^{n-1}$ (hint: use induction)
   (b) Show that $[x, f(p)] = i\hbar f'(p)$ for any function f that can be expanded in a power series
   (c) Evaluate $[x^2, p^2]$
   (d) Show that $e^{ipa/\hbar}\,|x_0\rangle$ is an eigenvector of x and find the corresponding eigenvalue.

5. Consider a Gaussian wavepacket $\psi(x) = ce^{\frac{-x^2}{2\sigma^2}}$, where c is an arbitrary constant that could be chosen so that the state is normalized. Calculate $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ in the quantum state described by the wavepacket. Show that the uncertainty relation $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4}$ is saturated.