Phys 251A Problem Set 3
Date posted: September 20, 2024
Due date: September 27, 2024

1. Solve the spin procession problem in the Heisenberg picture. The Hamiltonian is

$$H = \omega S_z$$

where ω is proportional to the magnetic field.

   (a) Write the Heisenberg equations of motion for $S_x(t)$ and $S_y(t)$ and solve them.

   (b) For a state in the $|\boldsymbol{n}+\rangle$ direction find the probability to measure $S_x = +\hbar/2$ at time t.

   (c) Find the expectation value of S_x at time t in the state $|\boldsymbol{n}+\rangle$.

   (d) Convince yourself that your answers to the two parts above make sense for various limiting values of $|\boldsymbol{n}\rangle$ (e.g. along the z axis or in the x-y plane).

2. Suppose that your TF made an error and wrote the Hamiltonian

$$H = H_{11} |1\rangle \langle 1| + H_{22} |2\rangle \langle 2| + H_{12} |1\rangle \langle 2|$$

where $|1\rangle$ and $|2\rangle$ are orthonormal. Simplify the time evolution operator and describe its action (hint: use power series, and you can assume $H_{11} = H_{22} = 0$ for simplicity). What fundamental principle of quantum mechanics is now violated, and why do its predictions no longer make sense?

3. Consider a free particle in one dimension, with $H = p^2/2m$. Evaluate $[x(t), x(0)]$ in the Heisenberg picture.

4. Compute the Heisenberg equations of motion for x and p for a particle with Hamiltonian $H = \frac{p^2}{2m} + V(x)$. Compare to the classical equations of motion, $F = ma$ and $p = mv$.

5. Consider two eigenstates $|a_1\rangle$ and $|a_2\rangle$ of a Hermitian operator A. Assume that the eigenvalues are distinct, $a_1 \neq a_2$. Suppose the Hamiltonian is

$$H = |a_1\rangle \delta \langle a_2| + |a_2\rangle \delta \langle a_1|$$

where δ is a real number.

   (a) Write down the energy eigenstates of the Hamiltonian and their energy eigenvalues

   (b) Suppose the state is $|a_1\rangle$ at time $t = 0$. Find the state at other times within the Schrodinger picture.

   (c) What is the probability of finding the state in $|a_2\rangle$ at time t if the state is $|a_1\rangle$ at time $t = 0$?

6. Consider a one dimensional simple harmonic oscillaor with annihilation operator

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right).$$

and the usual relationships $a |n\rangle = \sqrt{n} |n-1\rangle$ and $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$.

   (a) Evaluate $\langle m| x |n\rangle$, $\langle m| p |n\rangle$, $\langle m| \{x, p\} |n\rangle$, $\langle m| x^2 |n\rangle$, and $\langle m| p^2 |n\rangle$

   (b) Check that the quantum version of the "virial theorem" holds, i.e.

$$\left\langle \frac{p^2}{m} \right\rangle = \left\langle x \frac{dV}{dx} \right\rangle$$

   where the expectation value is taken in an energy eigenstate $|n\rangle$.

7. Consider two sets of simple harmonic oscillators described by annihilation operators $a_\pm$ and creation operators $a_\pm^\dagger$. Let us define

$$J_\pm = \hbar a_\pm^\dagger a_\mp, \quad J_z = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \quad N = a_+^\dagger a_+ + a_-^\dagger a_-$$

as well as

$$\boldsymbol{J}^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+).$$

Prove that

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [\boldsymbol{J}^2, J_z] = 0, \quad J^2 = \frac{\hbar^2}{2} N(N/2 + 1)$$