Phys 251A Problem Set 4
Date posted: September 27, 2024
Due date: October 4, 2024

1. Consider the coherent state of the one-dimensional simple harmonic oscillator

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle$$

where α is a complex number and $a^\dagger$ is the Harmonic oscillator creation operator

   (a) Show that $a|\alpha\rangle = \alpha|\alpha\rangle$ directly by acting on the above expression (hint: what is the value of the commutator $[a, f(a^\dagger)]$, for an arbitrary power-series-expandable f?)

   (b) Using the completeness property of coherent states, $1 = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha$, define a "coherent state wavefunction" $\psi(\alpha)$ and derive the action of $a^\dagger$ on $\psi(\alpha)$.

2. Again focusing on coherent states, consider the operator $T_\alpha = e^{\alpha a^\dagger - \overline{\alpha} a}$.

   (a) Show that T_α generalizes both boosts and translations (i.e. for which α does T_α correspond to a boost, and for which α does T_α correspond to a translation?).

   (b) Show that $T_\alpha|\beta\rangle$ is proportional to $|\beta + \alpha\rangle$. You may want to use Baker-Campbell-Hausdorff which yields a series expansion of Z in terms of X and Y where $e^Z = e^X e^Y$. Due to the oscillator algebra, the series truncates in this case: $Z = X + Y + \frac{1}{2}[X, Y] + \ldots$. You may also want to use the adjoint expansion $e^A B e^{-A} = e^{\mathrm{ad}_A} B$ and results from the problem (1)

   (c) From the previous part, conclude that $T_\alpha T_\beta = e^{i\phi} T_\beta T_\alpha$. Find ϕ in terms of α and β and interpret it in terms of an "area".

3. How many quantum mechanical states are there per unit volume for a free particle per unit energy? This object is usually called the "density of states" $g(E)$ such that the number of states in a large volume V in an energy window ΔE is

$$N = \int_{\Delta E} g(E) dE, \tag{1}$$

where N is typically proportional to volume V. In this problem we will compute $g(E)$ for a free particle, $H\psi(x, y, z) = \frac{-\hbar^2\nabla^2}{2m}\psi(x, y, z)$.

   (a) Let us consider a finite box of size $L \times L \times L$ with periodic boundary conditions: $\psi(x, y, z) = \psi(x+L, y, z) = \psi(x, y + L, z) = \psi(x, y, z + L)$. Find the energy eigenstates of H and their energies.

   (b) Consider a small window of energies dE, but even larger L so that $dE \gg 1/L$ and there are many states within the energy window dE. Compute the number of states dN that lie in the energy window dE as a function of L and E, and thus the density of states $g(E) = dN/dE$.

   (c) What is $g(E)$ for a particle in one dimension instead? Two dimensions?

4. Consider a Dirac particle in two dimensions. The Hamiltonian is

$$H = cp_x\sigma_x + cp_y\sigma_y + mc^2\sigma_z$$

where $\sigma_{x,y,z}$ are the Pauli matrices and satisfy $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and c is the speed of particles when $p \gg mc$.

   (a) Compute the energy eigenvalues and the energy eigenstates of H

   (b) Calculate the density of states $g(E) = dN/dE$ for $m = 0$

   (c) Calculate the density of states for generic m and compare your answer to an ordinary $p^2/2m$ particle in two-dimensions for $cp \ll mc^2$.