Phys 251A Problem Set 5
Date posted: October 4, 2024
Due date: October 11, 2024

1. Consider two Hamiltonians $H_1 = A^\dagger A$ and $H_2 = AA^\dagger$ for some operator A that is not necessarily Hermitian.
   (a) Show that H_1 and H_2 have a non-negative spectrum (all eigenvalues are non-negative).
   (b) Show that H_1 and H_2 have the same spectrum at nonzero energies by relating their eigenvectors.

2. In this problem we will use the tools developed in problem (1) solve the spectrum of following Hamiltonian exactly.

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{2V_0}{\cos^2(x/a)}, \qquad V_0 = \frac{\hbar^2}{2ma^2}$$

   Here the particle is confined to lie in the interval $-\pi a/2 < x < \pi a/2$. At the endpoints we impose $\psi(x = \pm\pi a/2) = 0$ in accordance with the fact that the potential is infinite at these points.
   (a) Rewrite the above problem in a non-dimensionalized form, with $y = x/a$ and $H_0 = H/V_0$.
   (b) Define the operator $A = \frac{d}{dy} + \tan y$ and compute the Hamiltonians $H_1 = A^\dagger A$ and $H_2 = AA^\dagger$. Recall that $\left(\frac{d}{dy}\right)^\dagger = -\frac{d}{dy}$.
   (c) One of H_1 and H_2 is particularly simple. Solve for its entire spectrum.
   (d) Show that the more complicated Hamiltonian has no zero energy states that satisfy the boundary condition (hint: use that $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$)
   (e) Use problem (1) of this problem set to deduce the entire spectrum of both $H_{1,2}$.
   (f) Find the eigenstates and eigenvalues of H_0 and its dimensionalized version H.

3. Consider a charged particle, in two dimensions, in a position independent magnetic field $B = \boldsymbol{\nabla} \times \boldsymbol{A} = \partial_x A_y - \partial_y A_x$. Note that $\boldsymbol{A}$ must be position dependent. The Hamiltonian is given by

$$H = \frac{\boldsymbol{\pi}^2}{2m}, \qquad \boldsymbol{\pi} = -i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}.$$

   where $\boldsymbol{\pi}$ is the "dynamical momentum" in the presence of the magnetic field.
   (a) Compute the Heisenberg equations of motion for x and y and physically justify the labeling of $\boldsymbol{\pi}$ as the "dynamical momentum."
   (b) Define a gauge transformation of an operator "M" as $M \to e^{-ie\phi(\boldsymbol{r})/\hbar} M e^{ie\phi(\boldsymbol{r})/\hbar}$, where M acts on wavefunctions $\psi(\boldsymbol{r})$. Show that, under a gauge transformation, the dynamical momentum $\boldsymbol{\pi}$ is replaced by a similar dynamical momentum except with $\boldsymbol{A}$ replaced by $\boldsymbol{A} - \boldsymbol{\nabla}\phi(\boldsymbol{r})$ (which does not change the magnetic field).
   (c) Compute $[\pi_x, \pi_y]$ in terms of the magnetic field B.
   (d) Compute the Heisenberg equations of motion for π_x and π_y and discuss them in the context of the classical Lorentz force law.

4. Charged particle in a field II
   (a) Compare the commutation relation $[\pi_x, \pi_y]$ to that of x and p, and the charged particle Hamiltonian to that of the Harmonic oscillator Hamiltonian. Write down ladder operators such that

$$H = \frac{\boldsymbol{\pi}^2}{2m} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).$$

   Compare to the so-called "cyclotron frequency" of classical mechanics. Eigenstates of $a^\dagger a$ here are called "Landau levels."

(b) It seems that we have gone from a continuum of states to a discrete set as soon as we have included a magnetic field. This is not the case, however, and each Landau level has many states inside it. To see this, we will construct operators that act "within" a Landau level. Consider the operators

$$R_x = x - \lambda\pi_y, \qquad R_y = y + \lambda\pi_x$$

where x and y are the usual position operators in two dimensions. Solve for λ such that $[R_x, \boldsymbol{\pi}] = 0$ and $[R_y, \boldsymbol{\pi}] = 0$. Conclude that $[R_{x,y}, a^\dagger a] = 0$, such that R_x and R_y act "within" each Landau level (they can be simultaneously diagonalized with the number operator).

(c) Compute $[R_x, R_y]$. Write down ladder operators b and $b^\dagger$ that each commute with both of a and $a^\dagger$. Show that there are infinitely many states within a Landau level.