Phys 251A Problem Set 7
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Throughout this problem set we will use the angular momentum algebra

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k, \qquad [J_z, J_\pm] = [J_z, J_x \pm iJ_y] = \pm\hbar J_\pm, \qquad J^2 = \sum_i J_i^2$$

where we will often work with sectors of fixed $J^2 = \hbar^2 j(j+1)$ because $[J^2, J_i] = 0$.

1. Beyond spin 1/2

   (a) Consider the angular momenta J_i for a spin 1 particle. The usual Hilbert space basis is given by the eigenstates of J_z, i.e. $|+\rangle = |j = 1, m = 1\rangle$ as well as $|0\rangle = |1, 0\rangle$ and $|-\rangle = |1, -1\rangle$. In this basis, using the angular momentum algebra, derive the explicit 3×3 matrices for $J_{x,y,z}$. These matrices generalize the matrices $\frac{\hbar}{2}\sigma_i$ that we used for spin 1/2.

   (b) For spin 1/2 particles we observed that the Pauli matrices further satisfied $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. For spin 1, are the anticommutators also proportional to the identity?

   (c) Show that, in the state $|j, m\rangle$, now for a general j, we have

$$\langle J_x\rangle = \langle J_y\rangle = 0, \qquad \langle J_x^2\rangle = \langle J_y^2\rangle = \frac{1}{2}\hbar^2(j(j+1) - m^2)$$

2. Consider the hydrogen atom Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}.$$

   We will write it as

$$H = \gamma + \frac{1}{2m}\sum_{k=1}^{3}\left(\hat{p}_k + i\beta\frac{\hat{x}_k}{r}\right)\left(\hat{p}_k - i\beta\frac{\hat{x}_k}{r}\right),$$

   where $\hat{p}_k$ and $\hat{x}_k$ are, respectively, the Cartesian components of the momentum and position operators, and β and γ are real constants to be adjusted so that the two Hamiltonians are the same.

   (a) Calculate the constants β and γ. Express them in terms of e^2 the Bohr radius a_0 and other constants.

   (b) Explain why for any state $\langle H\rangle \geq \gamma$. Find the wavefunction of the state for which this energy inequality is saturated. This is the ground state of Hydrogen. Give the normalized wavefunction.

3. Suppose you have a set of angular momentum operators $\hat{J}_i, i = 1, 2, 3$ that define an angular momentum $\hat{\mathbf{J}}$. A set of operators $\hat{W}_i$, with $i = 1, 2, 3$ is said to form a vector operator $\hat{\mathbf{W}}$ if

$$\left[\hat{J}_i, \hat{W}_j\right] = i\hbar\epsilon_{ijk}\hat{W}_k.$$

   Note that $\hat{\mathbf{J}}$ itself is a vector operator.

   (a) Show that when $\hat{\mathbf{J}}$ is taken to be the orbital angular momentum $\hat{\mathbf{L}}$ the position operator $\hat{\mathbf{x}}$ is a vector operator.

   (b) Show that if $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$ are vector operators, so is the cross product $\hat{\mathbf{U}} \times \hat{\mathbf{V}}$.

   (c) Show that if $\hat{\mathbf{W}}$ is a vector operator then

$$\left[\hat{\mathbf{J}}^2, \hat{\mathbf{W}}\right] = 2i\hbar(\hat{\mathbf{W}} \times \hat{\mathbf{J}} - i\hbar\hat{\mathbf{W}})$$

   Check that this formula holds when we choose $\hat{\mathbf{W}} = \hat{\mathbf{J}}$.

4. Consider a particle moving in the field of a magnetic monopole. We recall the commutation relations of the dynamical momentum, $\boldsymbol{\pi} = -i\hbar\boldsymbol{\nabla} - eA(\boldsymbol{r})$, and the magnetic field of the magnetic monopole

$$[\pi_i, \pi_j] = ie\hbar\varepsilon_{ijk}B_k(\boldsymbol{r}), \qquad B(\boldsymbol{r}) = \frac{g\boldsymbol{r}}{4\pi r^3}$$

where now $B(\boldsymbol{r})$ and r should be understood as functions of the vector of position operators $\boldsymbol{r}$. Since the monopole field is spherically symmetric, the angular momentum of a charge moving in a monopole field should be conserved. However the naive choice $\boldsymbol{L_0} = \boldsymbol{r} \times \boldsymbol{\pi}$ does not work (classically or quantum mechanically); it does not satisfy the angular momentum algebra and it does not commute with the Hamiltonian. There should be some generator of rotations, on physical grounds, so lets obtain it by guessing.

(a) Find $f(r)$ such that

$$[L_i, H] = 0, \qquad H = \frac{1}{2m}\boldsymbol{\pi}^2, \qquad \boldsymbol{L} = \boldsymbol{L}_0 + f(r)\hat{\boldsymbol{r}}.$$

It is possible to show that this same choice leads to L_i satisfying the angular momentum algebra, but you do not need to do this here.

(b) Unlike ordinary orbital angular momentum, we have found that $\boldsymbol{L}$ has a fixed component in the $\hat{\boldsymbol{r}}$ direction, proportional to the f that you found, that is not generated by the motion of the particle, that we are forced to consider. Compare this component to the Dirac quantization condition that we derived in lecture, $eg = 2\pi\hbar n$, and conclude that Dirac quantization is equivalent to angular momentum conservation. You should also conclude that the minimal monopole charge pair allowed by Dirac quantization has *half*-integer orbital angular momentum.