Phys 251A Problem Set 8

Date posted: November 8, 2024
Due date: November 15, 2024

1. True/False. Please briefly explain your reasoning.

   (a) The trace of angular momentum operators, Tr J_i, always vanishes

   (b) If J_{1i} and J_{2i} are angular momentum operators for two distinct particles "1" and "2" then the relative angular momenta, $J_{-i} = J_{1i} - J_{2i}$, also satisfies the angular momentum algebra.

   (c) If $[H, L_x] = 0$ and $[H, L_y] = 0$, where L_x and L_y are the orbital angular momenta in the x and y directions, then the Hamiltonian is rotationally symmetric in all directions.

   (d) There is a unitary operator R_j such that $R_j^\dagger J_i R_j = J_i$ for $i \neq j$ but $R_j^\dagger J_j R_j = -J_j$, where J_i are the angular momentum operators.

2. Consider three spin one particles, labeled 1,2, and 3, with spin operators $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$. The spins are placed on a circle and the interactions are between nearest neighbors. The Hamiltonian is

$$H = -\frac{\Delta}{\hbar^2}(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \boldsymbol{S}_2 \cdot \boldsymbol{S}_3 + \boldsymbol{S}_3 \cdot \boldsymbol{S}_1). \tag{1}$$

   You will find the total spin operator $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3$ useful, as well as the angular momentum decomposition $j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \ldots \oplus |j_1 - j_2|$.

   (a) What is the dimensionality of the space of the three combined particles?

   (b) Rewrite the Hamiltonian by expanding out $S^2 = \boldsymbol{S} \cdot \boldsymbol{S}$.

   (c) Find all the energy eigenvalues and their degeneracies

   (d) Write down the ground state of maximal S_z, in the form $|m_1, m_2, m_3\rangle$, as well as the ground state of next lowest S_z explicitly. Explain how you would find the other ground states as well, though for these a schematic form is sufficient.

3. Adding angular momenta (notational advice: if you find yourself getting confused labeling different types of states all with just explicit numbers, it might help to use $\pm, 0$ for spin 1 values of m, and similarly $\pm$ or $\uparrow\downarrow$ for spin 1/2 values of m, while reserving numbers for the total angular momentum states $|j, m\rangle$.)

   (a) Imagine a particle has orbital angular momentum $l = 1$ and spin $s = \frac{1}{2}$; what are the six states $|j, m\rangle$ in terms of the basis states $|m_l, m_s\rangle$? You can check your answer against Chebsh-Gordon coefficients looked up online, but you should derive this case yourself and show your work on your submission.

   (b) Consdier three spin 1/2 particles with basis states $|m_1, m_2, m_3\rangle$. Using the states we derived in lecture for adding two spin 1/2 particles, together with the previous part that adds a spin 1 particle to a spin half particle, find the possible values for the total angular momentum j and derive the eight states $|j, m\rangle$ for the entire system (you should not have to act with any operators, perhaps besides J_z where the actions is very straightforward, in this sub part.)